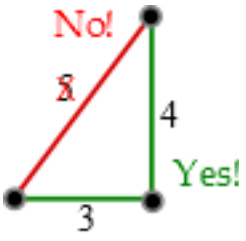
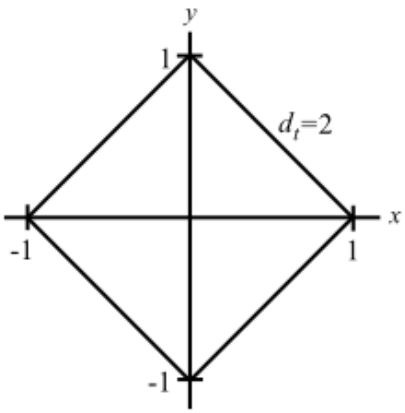
A Compendium

of

Taxicab Geometry

by Kevin P. Thompson





"No! Really? That's a *circle*?!"

1st Edition - 2025

Eratta

In a large work such as this mistakes are bound to occur. Please submit errata and questions to valleyforgepa@pm.me

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A Compendium of Taxicab Geometry

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Kevin P. Thompson

To Professor Tevian Dray
of Oregon State University
without whom I may never have become immersed
in
Taxicab Geometry

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Chapter 1
Preliminaries

Section 1.1: Introduction / Audience

The fundamental difference between Euclidean geometry and taxicab geometry is the metric (formula) used to measure distance. At first glance the impact of this change can easily be underestimated. A deeper investigation reveals changes that ripple through many, many fields of mathematics: geometry, angles and trigonometry, and even calculus. Often unconsciously, we apply the Euclidean metric in an almost uncountable number of places. When we stop and consider these usages, we discover that many of the formulae, truths, and concepts we are so familiar with change in taxicab geometry, sometimes dramatically.

The goal of this work is to reveal a broad array of the changes wrought throughout mathematics when the L^1 metric is used instead of the L^2 metric, or in layman's terms when the distance formula is changed from the familiar formula derived from the Pythagorean Theorem to one more suitable for a taxicab driving the streets of a perfect city.

Audience

Some topics addressed herein will be accessible to those with a standard secondary education. However, as indicated earlier, a change to the distance formula in a geometry has impacts in many more advanced fields such as trigonometry and calculus. For those topics, completion of a trigonometry course, a pre-calculus course, or the first year of a college calculus sequence will be necessary for the material to be accessible.

IMPORTANT NOTE

Please be sure to read Section 1.3 before beginning in-depth study of this book. The focus of this work is *pure taxicab geometry* and failure to realize this could cause much confusion at certain points, especially when comparing against other taxicab geometry writings out in the wild.

TAXICAB GEOMETRY RESEARCHERS

You are encouraged to declare in any of your research what version of taxicab geometry you are using - *pure* or *traditional / modified* based on the discussion in Section 1.3. This will help readers of your work understand how angles and other "extensions" of distance are being interpreted.

Section 1.2: Traditional Taxicab Geometry

At its simplest, traditional taxicab geometry changes the Euclidean distance formula to the metric proposed by Herman Minkowski where the distance between two points (x_1, y_1) and (x_2, y_2) is

$$d_t = |x_2 - x_1| + |y_2 - y_1| \tag{1.2.1}$$

The idea behind this distance formula, formally called a metric, is that the distance between two points is not measured on a straight line, but on horizontal and vertical lines as if traveling the streets of a perfect city. This is the origin of the name - taxicab geometry - since distance is measured by how a taxicab travels as opposed to how a bird flies.

This definition leaves other geometric features such as points, lines, and angles as Euclidean. Until 1996, this was the form in which the geometry was almost always investigated, discussed, and used. It was around this time that the author of this work and Rüstem Kaya independently began research into angles that natively belong to taxicab geometry thus launching investigations into a purer form of taxicab geometry.

Other Names

The taxicab metric often occurs in other contexts and has therefore been given various names over the years. The following are the most common names you may see for the taxicab metric in other contexts or fields of mathematics.

- rectilinear distance
- Manhattan distance
- the L^1 norm: the taxicab metric is another name for the norm of L^p space when $p = 1$:

$$\|x_p\| = \left(|x_1|^p + |x_2|^p \ldots + |x_n|^p \right)^{1/p}$$

Section 1.3: Pure Taxicab Geometry

As expressed in Section 1.2, much early research in taxicab geometry only took the idea halfway, in a sense. While the distance metric had been changed, its application throughout the geometry was incomplete. For instance, in Euclidean geometry, (radian) angles are defined as *distance* along the unit circle. Rather than apply this concept in taxicab geometry with the taxicab metric to come up with a taxicab version of angle measure, Euclidean angles and trigonometric were often retained. This produced a sometimes awkward mix of geometries.

While it is occasionally convenient to refer to Euclidean angles in taxicab geometry, this work will primarily deal with what should be called *pure taxicab geometry*, where any place the concept of distance is used the ***taxicab*** metric will be employed. This is in contrast to what should be called *modified taxicab geometry* (or perhaps *traditional taxicab geometry*) where taxicab geometry is mixed with Euclidean angles or some other non-taxicab concept.

Section 1.4: Euclid's Axioms

Euclidean geometry is an axiomatic system, meaning all features of the geometry can be derived from a small set of assumptions. These assumptions were first concisely expressed by the mathematician Euclid in his book *Elements* written around 300 B.C. When exploring non-Euclidean geometries, it is useful to determine which of the Euclidean axioms, or properties derived from these axioms, is no longer true. Since the time of Euclid, other axiomatic systems have been proposed that are equivalent to Euclid's but are more explicit and make fewer unstated assumptions. The axioms of David Hilbert (1862-1943) are an example.

One modern expression of basic properties of Euclidean geometry includes 13 properties:

1. Given any two points there is exactly one line containing them
2. Every line contains at least two points; the geometric space contains at least 3 non-collinear points
3. The distance between two points is a non-negative number and is zero only if the two points coincide
4. The distance from point A to point B equals the distance from B to A
5. The distance function satisfies the triangle inequality property
6. Given any line, there exists a one-to-one and onto function that acts as a ruler for all pairs of points A, B on the line
7. Each line divides the space into two convex half-planes
8. The angle measurement function assigns to each angle a real number between 0° and 180°
9. Given a ray $\overrightarrow{AB}$ on the edge of a half-plane and given a real number r between 0° and 180°, there is exactly one ray extending from A into the half-plane with the angle measure of r
10. The angle addition property
11. Two angles forming a straight angle sum to 180°
12. The side-angle-side (SAS) congruence property holds for triangles
13. The parallel postulate: Given a point A not on a line l, there is exactly one line through A parallel to l

Taxicab geometry only fails one of the axioms or postulates of Euclidean geometry, but it does so in grand style: property #12. In Euclidean geometry, if two sides and the included angle of two triangles are congruent, then the triangles are congruent. This is called the SAS (side-angle-side) property for triangles. (Other congruence properties are ASA and AAS.) This assumption does not hold in taxicab geometry, and one example can prove it fails dramatically. In Figure 1.4.1, two sides and all three angles of the two triangles are congruent (see Section 4.1 for more about taxicab angles). But, the third sides are not congruent. So, taxicab geometry not only fails the SAS property, but even an ASASA property does not guarantee congruence of triangles. (Note that this is the case for both *traditional* and *pure taxicab geometry.*)

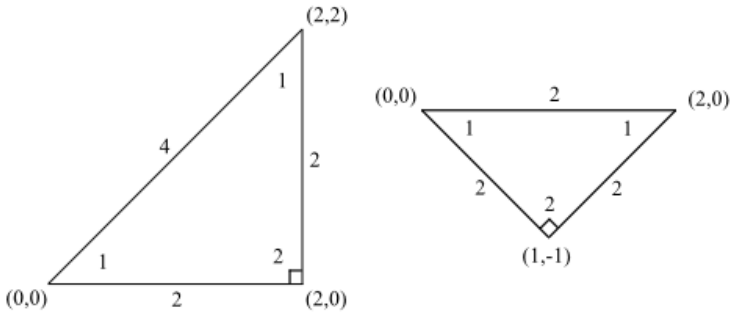


Figure 1.4.1 Two triangles that satisfy an ASASA property but are not congruent (distances are taxicab and angle measures are shown in t-radians, but the angles could be Euclidean and the example still holds).

In taxicab geometry, there is precisely one congruent triangle property. It relies on the fact that even in taxicab geometry, the sum of the angles of a triangle is always the same: 4 t-radians (the equivalent of 180° - see Section 6.1). The only congruent triangle property is SASAS: only if all three sides and two angles of two triangles are congruent will the triangles be congruent. This is explored more deeply in Chapter 6.

Chapter 2
The Taxicab Metric

Section 2.1: Distance Between Two Points

The heart of the contrast between Euclidean geometry and taxicab geometry is the change to the distance formula. As an example, the Euclidean distance between the points (1, 1) and (4, 5) is

$$\begin{aligned} d_e &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4-1)^2 + (5-1)^2} \\ &= \sqrt{(3)^2 + (4)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

However, the taxicab distance between the two points is

$$\begin{aligned} d &= |x_2 - x_1| + |y_2 - y_1| \\ &= |4 - 1| + |5 - 1| \\ &= |3| + |4| \\ &= 3 + 4 \\ &= 5 \end{aligned}$$

The difference between these two distance calculations is best seen visually. Figure 2.1.1 illustrates the difference. In Euclidean geometry, the distance between the points can be found by traveling along the line connecting the points. Inherently in the calculation you use the Pythagorean Theorem to compute the distance.

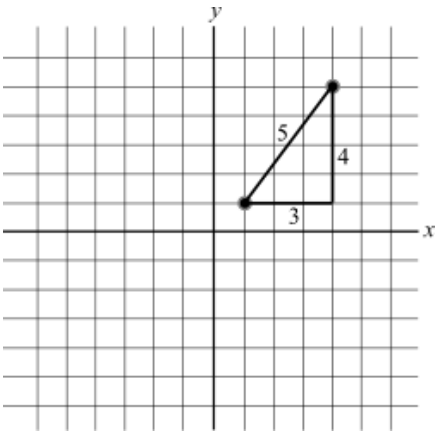


Figure 2.1.1 Visualizing distance in Euclidean and taxicab geometry. The Euclidean distance is 5; the taxicab distance is 3 + 4 = 7.

In taxicab geometry, the usual explanation is that to travel from one point to the next you must use only horizontal and vertical streets as if traveling by taxi in a perfect city. For the points given, the Euclidean distance according to the Pythagorean Theorem is 5 but the taxicab distance is simply the sum of the distances in the horizontal and vertical directions: 3 + 4 = 7.

In general, the only time you can expect the Euclidean and taxicab distance to agree is when both points lie along a horizontal or vertical line (Figure 2.1.2). This fact is proven more formally below.

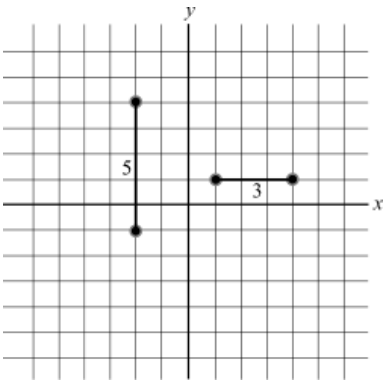


Figure 2.1.2 The Euclidean and taxicab distances are equal when both points lie along a horizontal or vertical line.

The City-Dweller's Shortest Distance

Anyone who has walked in a city center area where they are constrained to walk on sidewalks has likely noticed the concept of "closer" is different than normal. As an example, in Figure 2.1.3, as the crow flies the courthouse is closer to the post office (5 units) than the theater is (6 units). But, for one confined to walk the city streets, the theater is closer to the post office (6 units) than the courthouse is (7 units).

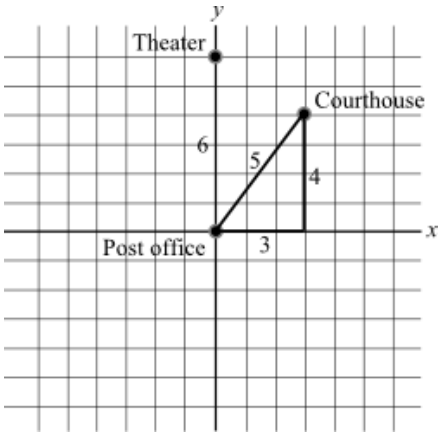


Figure 2.1.3 A city-dweller's view of shortest distance is different than a crow's: the theater is closer to the post office than the courthouse.

The Distance Impact

The concept of distance has far-reaching effects in a geometry. Everything from angles to conic sections to area and volume are in some manner dependent on the meaning of distance. Having changed the distance formula for Euclidean geometry and seen some of the basic effects, one should expect the ripple effect to continue deep into other concepts. The remainder of this book looks at different aspects of taxicab geometry to understand their nature and often contrast them with Euclidean geometry.

Converting Taxicab Distance to Euclidean Distance

At times it will be necessary to consider the length of line segment in an alternate geometry.

Theorem 2.1.1 *If m is the slope of the line between two points, then the Euclidean and taxicab distances between the points are related by the equation*

$$d_e = \frac{\sqrt{1+m^2}}{1+|m|} \cdot d \tag{2.1.1}$$

PROOF: We wish to find the expression for $f(m)$ such that $d_e = f(m) \cdot d$. Without loss of generality, consider the origin and the point (x, y):

$$\sqrt{x^2+y^2} = f(m) \cdot (|x| + |y|)$$

Dividing through by $|x|$ we get

$$\sqrt{1+m^2} = f(m) \cdot (1 + |m|)$$

where $m = y/x$ is the slope of the line connecting the points. Therefore,

$$f(m) = \frac{\sqrt{1+m^2}}{1+|m|}$$

and we have

$$d_e = \frac{\sqrt{1+m^2}}{1+|m|} \cdot d$$

□

This theorem makes it abundantly clear that the orientation / slope of a Euclidean line segment affects its taxicab length. Only when the correction factor equals 1 do the geometries agree on the length of a line segment. And, the correction factor is only equal to 1 when the slope of the line is $m = 0$ or as $m \to \infty$. This gives an algebraic justification of a fact that was already intuitively apparent.

Corollary 2.1.2 *The Euclidean and taxicab lengths of a line segment are only ever equal for horizontal and vertical lines.*

Euclidean trigonometric functions provide an alternative method for converting length between Euclidean and taxicab geometry. If in a situation like Figure 2.1.1 the Euclidean angle between the line segment and the horizontal is θ_e, then

$$d = d_e(|\cos_e \theta_e| + |\sin_e \theta_e|) \tag{2.1.2}$$

Transformations that Preserve Length

In Euclidean geometry translations, rotations, and reflections all preserve the length of a line segment. This is not the case in taxicab geometry. In Figure 2.1.1, if the diagonal line segment is rotated to be horizontal, it will have a shorter taxicab length (5) after the rotation. The following Theorem specifies the conditions for two line segments to have the same length in taxicab geometry.

Theorem 2.1.3 *Let A, B, C and D be four points in the Cartesian plane such that $A \neq B$ and $d_e(A,B) = d_e(C,D)$, and let m_1 and m_2 be the slopes of the lines $\overline{AB}$ and $\overline{CD}$, respectively.*

1. *If $m1 \neq m2$ are both non-zero, then $d(A,B) = d(C,D)$ if and only if $|m_1| = |m_2|$ or $|m1 \cdot m2| = 1$*
2. *If $m_1 = 0$ or $m_1 \to \infty$, then $d(A,B) = d(C,D)$ if and only if $m_2 = 0$ or $m_2 \to \infty$*

PROOF:

Case 1: Neither $\overline{AB}$ nor $\overline{CD}$ is vertical or horizontal. Let m_1 be the slope of $\overline{AB}$, m_2 be the slope of $\overline{CD}$.

$\Rightarrow$: Assume $d(A,B) = d(C,D)$. Then, using Theorem 2.1.1, since the Euclidean lengths of the line segments are equal,

$$\frac{\sqrt{1+m_1^2}}{1+|m_1|} = \frac{\sqrt{1+m_2^2}}{1+|m_2|}$$

$$\frac{\sqrt{1+m_1^2}}{\sqrt{1+m_2^2}} = \frac{1+|m_1|}{1+|m_2|}$$

$$\frac{1+m_1^2}{1+m_2^2} = \frac{1+2|m_1|+|m_1|^2}{1+2|m_2|+|m_2|^2}$$

$$1+2|m_2|+|m_2|^2+|m_1|^2+2|m_1|^2|m_2|+m_1^2m_2^2 = 1+2|m_1|+|m_1|^2+|m_2|^2+2|m_2|^2|m_1|+m_1^2m_2^2$$

$$|m_2|+|m_1|^2|m_2| = |m_1|+|m_2|^2|m_1|$$

$$|m_1|-|m_2|+|m_1||m_2|^2-|m_1|^2|m_2| = 0$$

$$(|m_1|-|m_2|)-|m_1||m_2|(|m_1|-|m_2|) = 0$$

$$(|m_1|-|m_2|)(1-|m_1m_2|) = 0$$

Therefore, either $|m_1| = |m_2|$ or $|m_1m_2| = 1$.

$\Leftarrow$: Assume $|m_1| = |m_2|$ or $|m1 \cdot m2| = 1$. In either case, reversing the steps in the work above leads to the factors $f_i(m)$ relating the lengths to be equal:

$$f_1(m) \cdot d(A, B) = f_2(m) \cdot d(C, d)$$

Therefore, $d(A, B) = d(C, D)$.

Case 2: Assume one line is either horizontal or vertical.

$\Rightarrow$: Assume $d(A, B) = d(C, D)$. Since the taxicab and Euclidean lengths are equal, $f(m_1) = f(m_2)$. If $\overline{AB}$ is horizontal or vertical, then the factor $f_1(m) = 1$ which means the factor $f_2(m) = 1$ which leads to either $m_2 = 0$ or $m_2 \to \infty$.

$\Leftarrow$: Assume the other line is horizontal or vertical. The the factors $f(m_i) = 1$ which means the taxicab distances are equal. $\square$

Using Theorem 2.1.3, we can summarize the operations on a line segment that will preserve its taxicab length. This will be an important tool used often in this book.

Corollary 2.1.4 *Aside from translation, the basic transformations of a line segment that preserve taxicab length are reflection about the lines* $y = x$, $y = -x$, $x = 0$, *or* $y = 0$.

When the transformations in Corollary 2.1.4 are composed, additional operations such as rotation by 1, 2, or 3 t-radians ($\pi/2$, π, or $3\pi/2$ radians) are seen to preserve taxicab length.

Section 2.2: Taxicab Circles

Having changed the Euclidean metric, even very simple geometric figures take on new forms. The circle is the poster child for taxicab geometric figures. Circles will be a central tool in many of the investigations we will pursue in future sections, so it is imperative that we establish a firm understanding of their form in taxicab geometry.

A circle is defined as the set of all points equidistant from a given point called the center of the circle. From the origin, to create a unit circle (a circle of radius 1), if we travel a distance $x < 1$ in the x-direction then we must travel a distance $y = 1 - x$ in the y-direction to have traveled a total distance of one unit. Therefore, points along this line are part of the unit circle centered at the origin. Following a similar process in the other three quadrants, we get Figure 2.2.1.

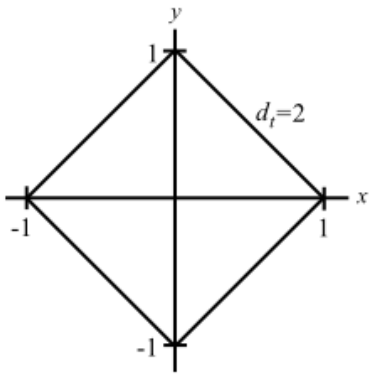


Figure 2.2.1 The taxicab unit circle

A circle with radius $r \neq 1$ will naturally be a similarly shaped "diamond" formed by the line $y = r - x$ and its counterparts in the other three quadrants.

An alternate way of exploring the shape of a taxicab circle involves analyzing how to move off of one of the points on the circle to another point while keeping the distance to the center constant. Therefore, begin with two points - the center and a point on what is to be the circle (Figure 2.2.2). If we begin moving from the point on the circle in a horizontal direction towards the center, the taxicab distance between the points will get smaller. To keep the distance constant, we must move in the vertical direction an equal amount to increase the distance and compensate for the horizontal movement. So, distance is kept constant by moving along lines of slope 1 or -1.

Extending this observation, we see that the set of all points equidistant from a central point is now a square with its edges oriented 45 degrees to the horizontal (Figure 2.2.1).

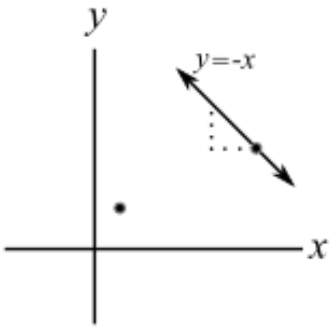


Figure 2.2.2 Finding a path in taxicab geometry to keep the distance to a fixed point constant.

The Equation of a Circle

For a circle of radius r centered at a point (h, k), any point (x, y) on the circle must be a distance r from the center. So, using the taxicab distance formula, the distance from (h, k) to a point (x, y) on the circle is

$$|x - h| + |y - k| = r \tag{2.2.1}$$

This is the general equation for a taxicab circle of radius r centered at (h, k).

Circles can alternatively be viewed as conics; we explore this approach in Section 7.2.

Circle-Point Theorems

In Euclidean geometry, we have the following results regarding the number of circles that can be drawn through a set of given points:

1. An infinite number of circles can be drawn through two given points
2. No circle can be drawn through three collinear points
3. Exactly one circle can be drawn through three non-collinear points

Taxicab geometry steadfastly continues to greatly complicate matters that are relatively simple in Euclidean geometry.

1. An infinite number of taxicab circles can be drawn through two given points. This is readily seen.
2. In most cases, no taxicab circle can be drawn through three collinear points. The exception involves points along a line of slope ± 1. In this case, the points lie along the side of any circle with side length longer than the greatest distance between the points.
3. If three points are non-collinear, there are 10 cases that describe whether a taxicab circle can be drawn through the points: two cases lead to no circle, three produce a unique circle, and five lead to an infinite number of circles. The cases largely break down into whether the three points when taken pairwise lie along shallow, steep, or diagonal lines. These were researched and documented by Çolakoğlu, and Kaya in their 2007 research paper "A synthetic approach to the taxicab circles." The concepts of midsets (bisectors) discussed in Section 2.4 is central to their investigation.

The scenario for three non-collinear points can be somewhat simplified using the concept of inscribed triangles. See Corollary 6.5.2 for a three-point circle theorem that helps distinguish the "no circle" cases from the rest.

Section 2.3: Distance to a Line

In Euclidean geometry, the distance between a point A and a line l is found by 1) constructing the line perpendicular to the original line that passes through the point, 2) identifying the point B of intersection of the two lines, and 3) measuring the distance between the original point A and the point of intersection B of the two lines (Figure 2.3.1).

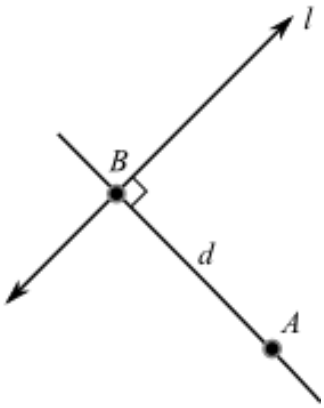


Figure 2.3.1 Distance from a point to a line in Euclidean geometry is measured along a line perpendicular to the line.

Another more visual approach is to inflate an Euclidean circle about the point until the circle just touches (is tangent to) the line (Figure 2.3.2). In this case, the distance to the line is the radius of the tangential circle. It is this approach we will use to conceptually explore the distance between a point and a line in taxicab geometry.

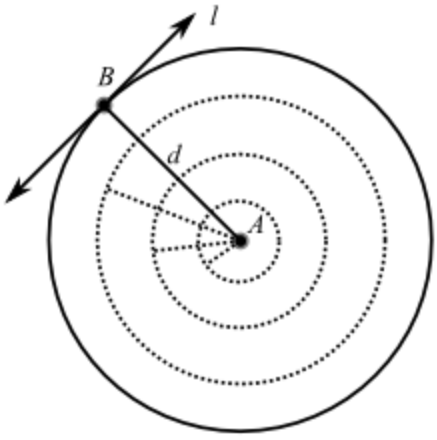


Figure 2.3.2 Visualizing Euclidean distance from a point to a line in terms of expanding circles about the point.

Consider the situation in Figure 2.3.3a. A taxicab circle is inflated about the point until the circle just touches the line. Notice that the distance from the point to the line is the direct vertical distance from the point to the line. But, perhaps this is due to the line being a shallow, or gradual line (a line whose slope has absolute value less than 1). What if the line was relatively steep? This situation is shown in Figure 2.3.3b where the slope of the line has absolute value greater than 1. The distance from the point to the line is now the direct *horizontal* distance from the point to the line. So, the position of the line has an impact on how the distance to a point is computed.

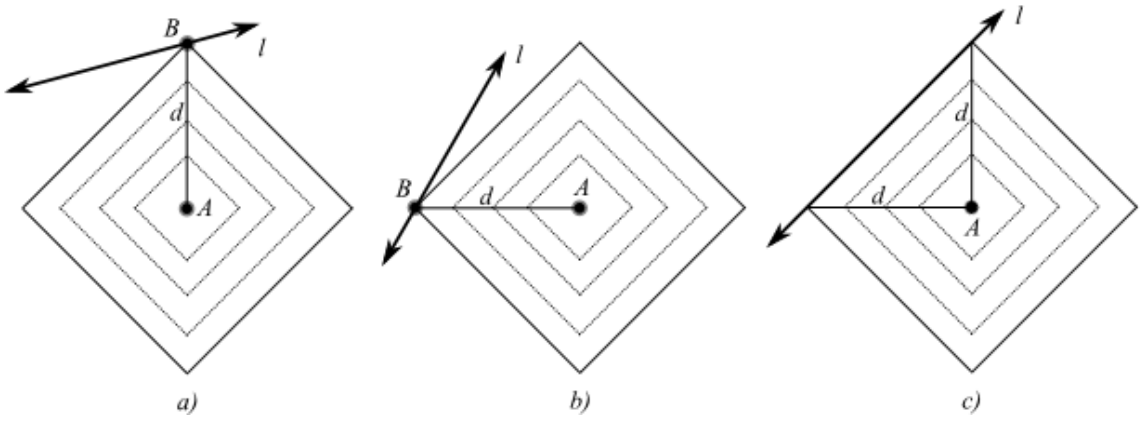


Figure 2.3.3 Visualizing taxicab distance from a point to a line in terms of expanding taxicab circles about the point. Three cases appear for a) shallow lines, b) steep lines, and c) perfectly diagonal lines.

To complete our analysis, consider a line of slope 1 (or -1) in Figure 2.3.3c. In this case, there are an infinite number of paths that can be followed to compute the distance from the point to the line because the inflated circle actually overlaps the line and does not just touch it at one point.

Just as the orientation of an Euclidean line segment affects its taxicab length, the position of a line affects the way in which distance is measured from a point to the line. The position of objects affecting their taxicab properties is a common theme in taxicab geometry. So, to summarize the cases for computing the distance between a point and a line:

- If the line is shallow (has slope with absolute value less than 1), measure the distance from the point to the line along a vertical line.
- If the line is steep (has slope with absolute value greater than 1), measure the distance from the point to the line along a horizontal line.
- If the line is diagonal (has slope with absolute value equal to 1), measure the distance from the point to the line along either a horizontal or a vertical line.

Theorem 2.3.1 The taxicab distance from a point $P = (x_0, y_0)$ to a line l of the form $ax + by + c = 0$ is

$$d(P, l) = \frac{|ax_0 + by_0 + c|}{\max\{|a|, |b|\}} \tag{2.3.1}$$

PROOF: The slope of the line is $m = -a/b$. For the shallow line case illustrated in Figure 2.3.3a, $|a| < |b|$ and the distance from the point P to the point on the line vertically above the point is

$$\begin{aligned} d &= |x_0 - x_0| + \left| -\frac{a}{b}x_0 - \frac{c}{b} - y_0 \right| \\ &= \frac{|-ax_0 - c - by_0|}{|b|} \\ &= \frac{|ax_0 + by_0 + c|}{|b|} \end{aligned}$$

Notice in the shallow case, the larger of $|a|$ and $|b|$ appears in the denominator. For the steep line case illustrated in Figure 2.3.3b, the distance is

$$\begin{aligned} d &= \left| -\frac{b}{a}y_0 - \frac{c}{a} - x_0 \right| + |y_0 - y_0| \\ &= \frac{|-by_0 - c - ax_0|}{|a|} \\ &= \frac{|ax_0 + by_0 + c|}{|a|} \end{aligned}$$

Again, the larger of $|a|$ and $|b|$ appears in the denominator. Therefore, we can combine these cases thus producing the desired formula:

$$d(P, l) = \frac{|ax_0 + by_0 + c|}{\max\{|a|, |b|\}}$$ □

Note that the line l in Theorem 2.3.1 has slope $m = -a/b$, so the selection of $|a|$ or $|b|$ by the max function in the denominator is simply a selection of whether the line is steep (large slope numerator - Figure 2.3.3b) or shallow (large slope denominator - Figure 2.3.3a).

Section 2.4: Midset of Two Points

In mathematics, a midset is the set of points that are all the same distance from two entities. In Euclidean geometry, the midset of two points is the perpendicular bisector of the line segment connecting the points. If we mimic the circle expansion strategy employed in the discussion about the distance from a point to a line (Section 2.3), we can easily see this (Figure 2.4.1).

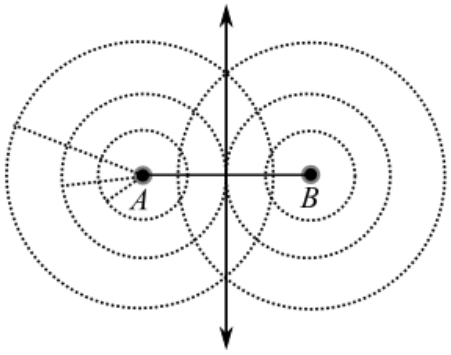


Figure 2.4.1 Forming the Euclidean midset (black vertical line) or perpendicular bisector of two points using expanding circles.

When this approach is employed in taxicab geometry, four distinct cases appear based on how expanding taxicab circles intersect. The first case shown in Figure 2.4.2 might easily lull you into thinking life has not changed much from Euclidean geometry. When the line connecting the points is horizontal or vertical, expanding taxicab circles about the points initially meet at a single point - the midpoint of the line segment connecting the points. As the circles continue to expand, the circles meet at exactly two points. This produces the same line as the Euclidean midset of the points - the Euclidean perpendicular bisector of the line segment connecting the points.

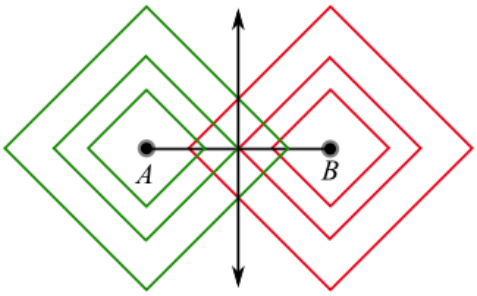


Figure 2.4.2 The taxicab midset (vertical black line) of two points lying along a horizontal line. In this case, the midset is still a perpendicular bisector. The illustration is similar for points on a vertical line.

If the line connecting the points is not horizontal or vertical, things begin to change. The expanding circles about the points initially meet in a line segment that passes through the midpoint of the line segment connecting the points (Figure 2.4.3). As the circles continue to expand, they meet at exactly two points. The expanding circles create two rays moving away from the original points. The midset is neither perpendicular to the segment $\overline{AB}$ nor even itself a straight line!

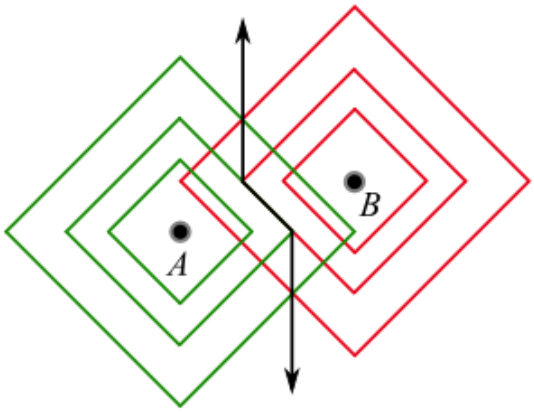


Figure 2.4.3 The taxicab midset (black line) of two points lying along a shallow line.

Note that this case actually breaks down into two cases. The rays formed by the intersections of the expanding circles have a different orientation depending on the slope of the line segment connecting the original points. Figure 2.4.4 shows a line segment with a slope of magnitude greater than 1 whereas Figure 2.4.3 illustrated a line segment with slope of magnitude less than 1.

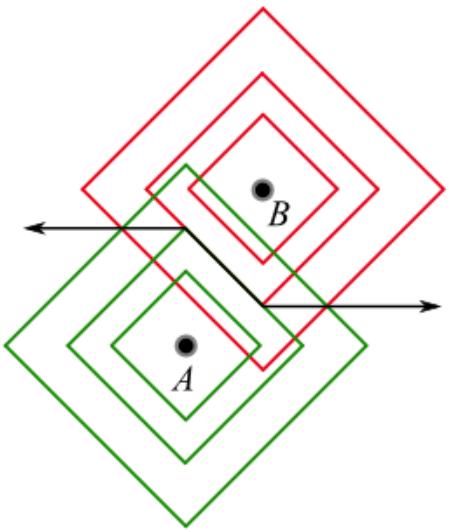


Figure 2.4.4 The taxicab midset (black line) of two points lying along a steep line.

This leaves only the case where the slope of the line connecting the points has absolute magnitude 1 (Figure 2.4.5). In this case the expanding circles also initially meet in a line segment that is actually perpendicular to the segment $\overline{AB}$. But, as the expansion continues, the circles meet in line segments instead of points. This creates a midset consisting of two regions of points joined by a diagonal line.

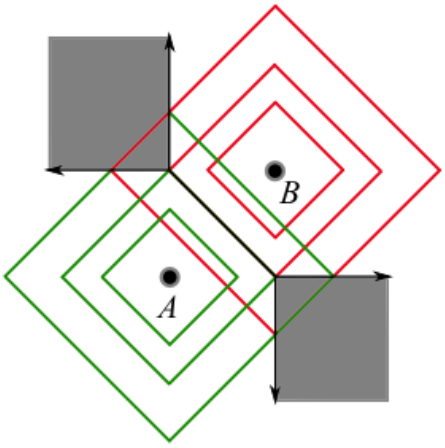


Figure 2.4.5 The taxicab midset (black lines *and* gray shaded regions) of two points lying along a diagonal line.

Only when the line segment connecting the points is horizontal or vertical is the midset of the two points a simple line like the Euclidean midset of two points. In all other cases the midset is at best a piecewise-defined line and at worst not a line at all! As in Euclidean geometry, the taxicab midset always includes the midpoint of the line segment connecting the points.

Section 2.5: Advanced Real Analysis Topics

The taxicab distance formula can be examined in the context of several advanced mathematical topics such as metric spaces, norms, and inner products.

Definition 2.5.1 *The taxicab measure of a point $x \in \mathbb{R}^n$ is*

$$t(\mathbf{x}) = \sum_{i=1}^{n} |x_i|$$

Theorem 2.5.1 *Taxicab measure is a norm on the vector space $\mathbb{R}^n$.*

PROOF: We must check the 3 conditions for a norm.

1) $\|\mathbf{x}\| = t(\mathbf{x}) = \sum_{i=1}^{n} |x_i| = 0$ if and only if each of the $x_i = 0$, meaning $\mathbf{x} = (0,0,...,0)$
2) $\|\mathbf{x}+\mathbf{y}\| = \sum_{i=1}^{n} |x_i + y_i| \le \sum_{i=1}^{n} |x_i| + |y_i| = \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|\mathbf{x}\| + \|\mathbf{y}\|$ by the triangle inequality for absolute value.
3) $\|c\,\mathbf{x}\| = \sum_{i=1}^{n} |c\,x_i| = \sum_{i=1}^{n} c\,|x_i| = c\sum_{i=1}^{n} |x_i| = c\|\mathbf{x}\|$ □

A normed vector space is automatically a metric space using the metric

$$d(x,y) = \|x - y\| = \sum_{i=1}^{n} |x_i - y_i|$$

but we can check the details for good measure (no pun intended).

Theorem 2.5.2 *The set $\mathbb{R}$ coupled with the taxicab distance formula $d(x,y) = |x-y|$ for $x, y \in \mathbb{R}$ is a metric space.*

PROOF: We must check 4 conditions on the function d claiming to be a metric. Let $x, y, z \in \mathbb{R}$.

1) $d(x,y) \ge 0$. This is true by the definition of absolute value.
2) $d(x,y) = 0$ if and only if $x = y$. An absolute value is zero if and only if the inside is zero, which only occurs when $x = y$.
3) $d(y,x) = |y-x| = |-(x-y)| = |x-y| = d(x,y)$
4) $d(x,y) = |x-y| = |(x-z)+(z-y)| \le |x-z| + |z-y| = d(x,z) + d(z,y)$ by the triangle inequality for absolute value. □

Corollary 2.5.3 *The set $\mathbb{R}^n$ coupled with the taxicab distance formula*

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

for $x, y \in \mathbb{R}^n$ is a metric space.

We now wish to examine whether the taxicab metric space is an inner product space and correct a significant misconception.

Neither the definition of an inner product (space) in general nor the "usual" inner product in Euclidean geometry mentions a norm, a metric, or distance in any form or fashion. An inner product is a particular kind of function $f : V \times V \to \mathbb{R}$ on a vector space whose definition is not dependent on the vector space being a normed vector space or a metric space.

Since taxicab geometry and Euclidean geometry are defined over the same vector space, the usual Euclidean inner product is valid in taxicab geometry. Therefore, taxicab geometry is an inner product space. Period.

When it is said that "taxicab geometry is not an inner product space" what is meant is that it is not *a particular kind* of inner product space. Specifically, taxicab geometry is not an inner product space where the taxicab norm and the inner product satisfy the relationship $\|v\| = \sqrt{\langle v, v \rangle}$. In this sense, taxicab geometry is not the *usual* or *preferred* kind of inner product space. The relationship between the taxicab norm and the inner product is not the normal, natural relationship we are accustomed to.

To investigate this further, let's explore the usual argument for why taxicab geometry is not an inner product space, or to say it correctly, why it is not the preferred kind of inner product space.

The traditional argument goes as follows. If the norm of a metric space satisfies the Parallelogram Law, then there is an inner product associated with the norm. In other words, if a norm satisfies this identity, it implies that an inner product can be defined from it using what is called the Polarization Identity:

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right)$$

Given a parallelogram $ABCD$ (Figure 2.5.1), the Parallelogram Law states

$$\|u + v\|^2 + \|u - v\|^2 = 2 \left(\|u\|^2 + \|v\|^2 \right)$$

$$2l^2 + 2w^2 = d_1^2 + d_2^2$$

But, taxicab geometry does not satisfy the Parallelogram Law. In Figure 2.5.2, twice the sum of the squares of the sides (122) is not equal to the sum of the squares of the diagonals (202). (However, we will explore a limited parallelogram type property for taxicab geometry in Section 5.5.) Therefore, the taxicab metric space is not also a (natural) inner product space.

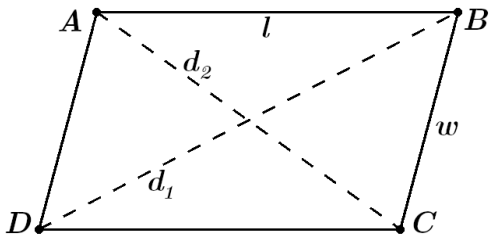


Figure 2.5.1 The Euclidean Parallelogram Law

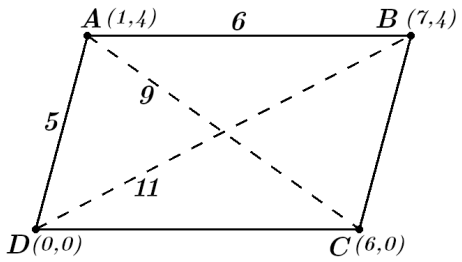


Figure 2.5.2 Taxicab geometry does not satisfy the Parallelogram Law

But, there is a catch. There is a hidden assumption in the logic above. Let's more carefully state the theorems being used.

Theorem Let V be a vector space over $\mathbb{R}$ and $\|\cdot\| : V \to \mathbb{R}$ a norm on V that satisfies the Parallelogram Law $\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$ for each $u, v \in V$. Then the function $\langle u, v\rangle : V \times V \to \mathbb{R}$ defined by $\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$ is an inner product on V and the norm satisfies $\|v\| = \sqrt{\langle v, v\rangle}$.

Corollary Let $\left(V, \|\cdot\|\right)$ be a normed vector space over $\mathbb{R}$. Then there exists an inner product $\langle\,\cdot\,,\cdot\,\rangle : V \times V \to \mathbb{R}$ such that $\|v\| = \sqrt{\langle v, v\rangle}$ if and only if $\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$ for each $u, v \in V$.

The key phrase is "such that $\|v\| = \sqrt{\langle v, v\rangle}$" in the Corollary. The Parallelogram Law is linked to the existence of an inner product *of a particular kind*, one such that $\|v\| = \sqrt{\langle v, v\rangle}$, meaning the inner product of a vector with itself is equal to the square of the length of the vector. So, it is true that taxicab geometry is not an inner product space in which the inner product is related to the norm by this relationship. But, that doesn't mean that a different kind of inner product couldn't be defined on the space which induces the taxicab norm. And, it doesn't meant that the usual Euclidean inner product can't be retained but the norm and metric are induced via a different relationship. This is precisely what Ekici, Kocayusufoğlu, and Akça did in 1998, although their research is very confused and muddled at the beginning.*

With that confusion now sorted out, the challenge at present is to derive the taxicab norm we expect from this inner product since it is known not to occur with the traditional relationship $\|v\| = \sqrt{\langle v, v\rangle}$. Ekici, et. al provide us the answer as the most valuable nugget of their research paper.

Theorem 2.5.4 *For a vector $v = (v_1, v_2) \in \mathbb{R}^2$ the taxicab norm can be induced from the usual $\mathbb{R}^2$ inner product $\langle\,\cdot\,,\cdot\,\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ using*

$$\|v\| = \sqrt{\langle v, v\rangle + 2\,|\,v_1 v_2\,|}$$

PROOF:

$$\begin{aligned}
\sqrt{\langle v, v\rangle + 2\,|\,v_1 v_2\,|} &= \sqrt{v_1^2 + v_2^2 + 2\,|\,v_1 v_2\,|} \\
&= \sqrt{\left(\,|\,v_1\,| + |\,v_2\,|\,\right)^2} \\
&= |\,v_1\,| + |\,v_2\,| \\
&= \|v\|
\end{aligned}$$

□

---

* The inner product given by Ekici, et. al is actually the same as the usual Euclidean inner product. I am not sure why it is defined in the manner they chose. The sign choices in the different cases match the natural result of the multiplication that is occurring, so the cases and conditions are simply unnecessary. In the end it would seem that about half of their research paper was unnecessary.

A major consequence of the change in relationship between the inner product and the taxicab norm shows up in the geometric interpretation of the inner product as being related to the angle between the two vectors. Surprisingly, the relationship still holds up when the vectors are in neighboring quadrants. But, when the vectors are in the same quadrant or "opposite" quadrants, an algebraic correction factor is needed and the relationship is lost.

Theorem 2.5.5 *For two vectors $u, v \in \mathbb{R}^2$ with angle θ between them, if u and v are in neighboring quadrants, then $\langle u, v\rangle = \|u\|\|v\| \cos\theta$.*

To summarize,

- Can an inner product be defined in taxicab geometry? Yes. Taxicab geometry uses the same underlying vector space as Euclidean geometry, so the usual Euclidean inner product applies in taxicab geometry as well.
- Does the usual Euclidean inner product induce the taxicab norm in the preferred way so that $\|v\| = \sqrt{\langle v, v\rangle}$? No.
- Is there a different inner product that could be defined on the taxicab vector space so that the inner product induces the taxicab norm in the preferred way? No.
- Does the usual Euclidean inner product induce the taxicab norm in a way different than the preferred way? Yes. The usual Euclidean inner product is related to the taxicab norm via the relationship

$$\|v\| = \sqrt{\langle v, v\rangle + 2\,|\,v_1 v_2\,|}$$

Note that in $\mathbb{R}^3$ the inner product to taxicab norm relationship is

$$\|v\| = \sqrt{\langle v, v\rangle + 2\,|\,v_1 v_2\,| + 2\,|\,v_1 v_3\,| + 2\,|\,v_2 v_3\,|}$$

Chapter 3
The Nature of Length, Area, and Volume

Section 3.1: The Nature of Length

The first dimensional measure we will examine is the simplest of the measures: the measurement of length. This is also, for line segments at least, the most well understood measure in taxicab geometry.

One-dimensional Length

The simplest measurement of length is the length of a line segment in one dimension. On this point, Euclidean and taxicab geometry are in complete agreement. The length of a line segment from a point $A = a$ to another point $B = b$ is simply the number of unit lengths covered by the line segment,

$$d_e(A, B) = d_t(A, B) = |b - a|$$

Two-dimensional Linear Length

In two dimensions, however, the Euclidean and taxicab metrics are not always in agreement on the length of a line segment. For line segments parallel to a coordinate axis, such as the line segment with endpoints $A = (x_1, y)$ and $B = (x_2, y)$, there is agreement since both metrics reduce to one-dimensional measurement: $d_e(A, B) = d_t(A, B) = |x_2 - x_1|$. Only when the line segment is not parallel to one of the coordinate axes do we finally see disagreement between the Euclidean and taxicab metrics. The taxicab length of such a line segment can be viewed as the sum of the Euclidean lengths of the projections of the line segment onto the coordinate axes (Figure 3.1.1), a restatement of Equation 2.1.2:

$$d_t = d_e |\cos\theta_e| + d_e |\sin\theta_e| \tag{3.1.1}$$

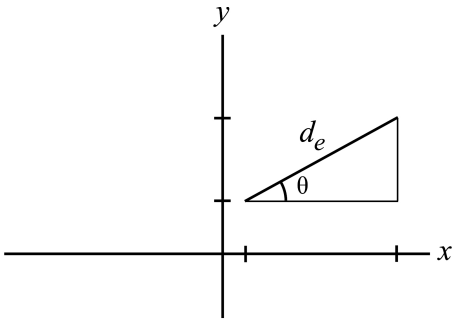


Figure 3.1.1 Taxicab length as a sum of Euclidean projections onto the coordinate axes.

The Pythagorean Theorem tells us the Euclidean and taxicab lengths will generally not agree for line segments that are not parallel to one of the coordinate axes. Line segments of the same Euclidean length will have various taxicab lengths as their position relative to the

coordinate axes changes. If one were to place a scale on a slanted line, the Euclidean and taxicab markings would differ with the largest discrepancy being at a 45° angle to the coordinate axes (Figure 3.1.2).

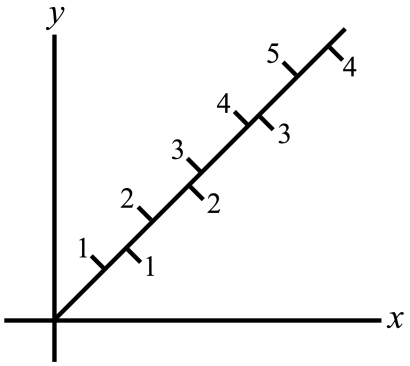


Figure 3.1.2 The scale of measurement differs along a slanted line for Euclidean geometry (below the line) and taxicab geometry (above the line).

These cases and types of length measurement are well known and are well understood to those familiar with taxicab geometry. But, even in two dimensions, there is at least one other type of length measurement in Euclidean geometry: the length of a curve. How is the arc length of a (functional) curve in two (or three) dimensions measured in taxicab geometry? We will cover this topic in Section 8.1.

Three-dimensional Linear Length

The taxicab length of a line segment in three dimensions is a natural extension of the formula in two dimensions. For a line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2),

$$d_t = |x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1|$$

There are no surprises here, and the taxicab length can be viewed as the sum of the Euclidean lengths of the projections of the line segment onto the three coordinate axes. In addition, the Euclidean and taxicab lengths of a line segment in three dimensions will only agree when the line segment lies in a plane parallel to one of the xy-plane, xz-plane, or yz-plane *and* the segment is parallel to one of the coordinate axes. These restrictions cause two of the three coordinate differences in the Euclidean distance formula to be zero thus allowing the square root to be applied to the square of the difference in the third coordinate thus producing an expression equivalent to the taxicab distance.

Conclusion

The primary item to note here is that Euclidean and taxicab geometry agree on the length of a line segment in one dimension but often disagree on the length of a line segment in higher dimensions. This is the first in what will become a pattern as we continue this chapter.

Section 3.2: The Nature of Area

In much early research regarding area in taxicab geometry, the underlying assumption had been that the area of a figure should agree in Euclidean and taxicab geometry, although the formulae to compute the area could be vastly different. Additionally, most investigations only examined area in two dimensions. In this section we wish to examine this assumption about area and explore surface area in three dimensions.

Figure 3.2.1 gets right to the question of what area means in taxicab geometry. The figures shown are each squares in both geometries: all sides and angles have the same measure. If we cling to the notion that the area of a square is the square of the length of its sides, these figures have equal Euclidean area (2) but very different taxicab areas (4 and 2). The position of the sides of the square relative to the coordinate axes in taxicab geometry affects their length which in turn affects the area calculation when clinging to the formula $A = s^2$.

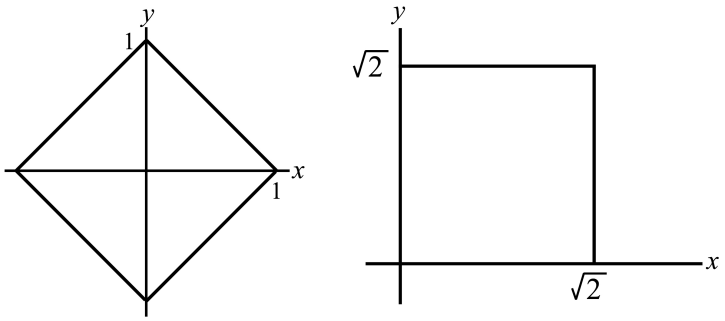


Figure 3.2.1 Squares with equal Euclidean area (2) but different taxicab area (4 and 2, respectively) when viewing area as the square of the length of the sides.

We have a choice before us. We can cling to the area of a square being the square of its sides and live with the position of a figure affecting its taxicab area just as the position of a line segment affects its taxicab length. Or, we can maintain consistency with the Euclidean area of the figure and, as in this example, seek a new formula for the area of a square in taxicab geometry. Several points here and later in this paper will hopefully help us to make a firm decision here.

Reviewing the nature of length in taxicab geometry in Section 3.1, we see agreement with Euclidean geometry on the length of a line segment (a one-dimensional "figure") in one dimension. Only for line segments in two dimensions, a dimension higher than the figure, do we see discrepancies between the geometries.

If we apply this same logic to area, the geometries should agree on area (a two-dimensional figure) in two dimensions. Only for the area of a surface in three dimensions would we expect to see differences because the figure would have a different position relative to the coordinate planes.

In addition, the area of a figure has traditionally been viewed as the number of square units of the plane enclosed by the figure. Since Euclidean and taxicab geometry are built on the Cartesian coordinate system, it seems logical the computation of area in two dimensions should agree in these two geometries.

With this reasoning, we proceed with the traditional Euclidean concept of area in two dimensions while looking for the proper extension of the concept to three dimensions. Therefore, for our example, we would need a new formula for the taxicab area of a square (see Section 5.4).

Surface Area in Three Dimensions

As stated above, we should expect the taxicab area of a "flat" surface in three dimensions to differ from the area of the "same" figure (in a Euclidean sense) in one of the coordinate planes. This follows from the pattern seen with line segments in one and two dimensions. To discover how the area will differ, consider a plane rotated upward from the xy-plane along the y-axis (Figure 3.2.2).

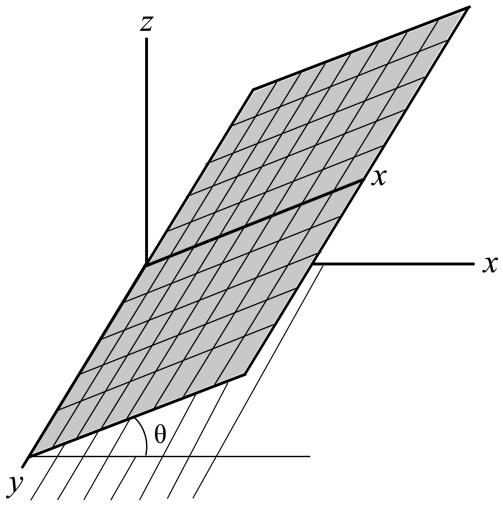


Figure 3.2.2 A plane rotated up from the xy-plane along the y-axis will have different Euclidean and taxicab scales of measurement in the x-direction.

In a manner similar to Figure 3.1.2, the taxicab scale in the x-direction will be compressed in the rotated plane compared to the Euclidean scale. The angle θ formed in a plane parallel to the xz-plane will determine the scaling effect seen in the one dimension of the rotated plane with the maximal difference occurring at 45°. The result is the "same" figure (in a Euclidean sense) in the xy-plane and in the rotated plane will have different taxicab areas. (Interestingly, the taxicab coordinate grid in the rotated plane is not even composed of Euclidean squares but of Euclidean rectangles since the taxicab scale in the y-direction is unaffected by the specific rotation performed.)

As a specific example, a unit square at the origin in the xy-plane will have a Euclidean and taxicab area of 1. If we rotate the square upward 45° along the y-axis, the Euclidean area will be unchanged but the taxicab area will now be $\sqrt{2}$ since by application of Equation 3.1.1 one side will now have taxicab length $d = \cos_e 45^\circ + \sin_e 45^\circ = \sqrt{2}$.

If the plane is instead rotated along the x-axis, the taxicab area is again found by scaling the Euclidean area. When rotation occurs in both manners, the scaling factors compound each other. So, the taxicab area of a figure in the rotated plane can be found from its Euclidean area using Equation 3.1.1 as a guide for each direction of rotation: the scaling factors are sums of the (Euclidean) cosine and sine of the angles α_e and β_e formed by lines in cross-sectional planes parallel to the xz-plane and the yz-plane similar to the single rotation illustrated in Figure 3.2.2,

$$A_t = A_e \left(|\cos_e \alpha_e| + |\sin_e \alpha_e| \right) \left(|\cos_e \beta_e| + |\sin_e \beta_e| \right) \tag{3.2.1}$$

Section 3.3: The Nature of Volume

If we follow the same pattern used for length and area measures, we expect the Euclidean and taxicab volume of a solid (a three-dimensional object) in three dimensions to be equal. Only in four dimensions would we expect orientation of the solid to impact the taxicab volume when compared to the Euclidean volume. We will not undertake any investigations in this text beyond three dimensions. Our primary analysis of volume will be with respect to solids of revolution in Section 8.4.

Conclusion

Generalizing the patterns we have set forth in this chapter, it appears that the taxicab measure of a n-dimensional figure in n-dimensional space will agree with the Euclidean measure of the figure. However, in $(n+1)$-dimensional space, the taxicab measure of the figure will in general depend on its position in the space when compared with the Euclidean measure.

Chapter 4
Angles and Trigonometry

Section 4.1: Taxicab Angles and π_t

There are at least two common ways of defining angle measurement: in terms of an inner product and in terms of the unit circle. For Euclidean space, these definitions agree. However, the taxicab metric is not an inner product since the natural norm derived from the metric does not satisfy the parallelogram law. Thus, we will define angle measurement on the unit taxicab circle which is shown in Figure 2.2.1.

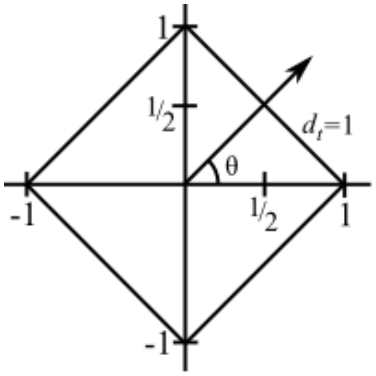


Figure 2.2.1 The taxicab unit circle and a 1 t-radian angle

Definition 4.1.1 *A **t-radian** is an angle whose vertex is the center of a unit (taxicab) circle and intercepts an arc of length 1. The taxicab measure of a taxicab angle θ is the number of t-radians subtended by the angle on the unit taxicab circle about the vertex.*

It follows immediately that a taxicab unit circle has 8 t-radians since the taxicab unit circle has a circumference of 8. For reference purposes the Euclidean angles $\pi/4$, $\pi/2$, and π in standard position now have measure 1, 2, and 4, respectively. The following theorem gives the formula for determining the taxicab measures of some other Euclidean angles.

Theorem 4.1.1 *An acute Euclidean angle ϕ_e in standard position has a taxicab measure of*

$$\theta = 2 - \frac{2}{1 + \tan_e \phi_e} = \frac{2 \sin_e \phi_e}{\sin_e \phi_e + \cos_e \phi_e} \tag{4.1.1}$$

PROOF: The taxicab measure θ of the Euclidean angle ϕ_e is equal to the taxicab distance from $(1, 0)$ to the intersection of the lines $y = -x + 1$ and $y = x \tan_e \phi_e$ at point P. The x-coordinate of this intersection is

$$x_0 = \frac{1}{1 + \tan_e \phi_e}$$

and thus the y-coordinate of P is $y_0 = -x_0 + 1$. Hence, the taxicab distance from $(1, 0)$ to P is

$$\theta = 1 - x_0 + y_0 = 2 - \frac{2}{1 + \tan_e \phi_e}$$

□

Definition 4.1.2 *The **reference angle** of an angle ϕ is the smallest angle between ϕ and the x-axis.*

Theorem 4.1.1 can easily be extended to any acute angle lying entirely in a quadrant.

Corollary 4.1.2 *If an acute Euclidean angle ϕ_e with Euclidean reference angle ψ_e is contained entirely in a quadrant, then the angle has a taxicab measure of*

$$\begin{aligned}\theta &= \frac{2}{1+\tan_e \psi_e} - \frac{2}{1+\tan_e(\phi_e+\psi_e)} \\ &= \frac{2\sin_e \phi_e}{(\cos_e(\phi_e+\psi_e)+\sin_e(\phi_e+\psi_e))(\cos_e \psi_e+\sin_e \psi_e)}\end{aligned} \tag{4.1.2}$$

This corollary implies the taxicab measure of a Euclidean angle in non-standard position is not necessarily equal to the taxicab measure of the same Euclidean angle in standard position. Thus, although angles are translation invariant, they are not rotation invariant. This is an important consideration when dealing with triangles in taxicab geometry.

In Euclidean geometry, a device such as a cross staff or sextant can be used to measure the angular separation between two objects. The characteristics of a similar device to measure taxicab angles would be very strange to inhabitants of a Euclidean geometry; measuring the taxicab size of the same Euclidean angle in different directions would usually yield different results. Thus, to a Euclidean observer the taxicab angle measuring device must fundamentally change as it is pointed in different directions. Of course this is very odd to us since our own angle measuring devices do not appear to change as we point them in different directions.

The taxicab measure of other Euclidean angles can also be found. Except for a few cases, these formulas will be more complicated since angles lying in two or more quadrants encompass corners of the unit circle.

Lemma 4.1.3 *The taxicab measure of any Euclidean right angle is 2 t-radians.*

PROOF: Without loss of generality, let θ be an angle encompassing the positive y-axis. As shown in Figure 4.1.2, split θ into two Euclidean angles α_e and β_e with reference angles $\pi/2-\alpha_e$ and $\pi/2-\beta_e$, respectively. Using Theorem 4.1.1, we see that

$$\begin{aligned}\theta &= \frac{2\sin_e \alpha_e}{\cos_e \alpha_e+\sin_e \alpha_e} + \frac{2\sin_e \beta_e}{\cos_e \beta_e+\sin_e \beta_e} \\ &= \frac{2\sin_e \alpha_e}{\cos_e \alpha_e+\sin_e \alpha_e} + \frac{2\cos_e \alpha_e}{\sin_e \alpha_e+\cos_e \alpha_e} \\ &= 2\end{aligned}$$

since $\alpha_e+\beta_e=\pi/2$. □

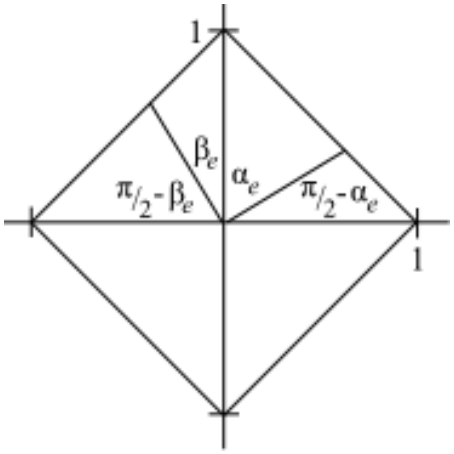


Figure 4.1.2 Taxicab right angles are precisely Euclidean right angles.

We now state the taxicab version of the familiar result for the length of an arc from Euclidean geometry and note that its proof is obvious since all distances along a taxicab circle are scaled equally as the radius is changed. This result will be used when we turn to congruent triangle relations (Section 6.2) and the concept of parallax (Section 9.1).

Theorem 4.1.4 (Arc Length Theorem) *The length s of the arc intercepted on a (taxicab) circle of radius r by the central angle with taxicab measure θ is given by $s = r\theta$.*

From the previous theorem we can easily deduce the following result.

Corollary 4.1.5 *Every taxicab circle has $8 = 2\pi_t$ t-radians.*

It should also be noted that the ratio of the circumference of a taxicab circle to its diameter is constant just as it is in Euclidean geometry. For a taxicab circle of radius r, the circumference is $8r$ and its diameter is $2r$ which gives a ratio of 4. This constant in Euclidean geometry is of course given the label of π and has an irrational value approximately equal to 3.14159265... In taxicab geometry the value of π is an integer: $\pi_t = 4$. (There is an interesting research paper called "The π's Go Full Circle" by Russell Euler that investigates the value of π in the various L^p spaces.) Also note that the formula for the circumference of a taxicab circle is very familiar but of course adapted for the change in value of π:

$$C = 8r = 2\pi_t r \tag{4.1.3}$$

.

Section 4.2: The (Missing) Inscribed Angle Theorem

In Euclidean geometry, using any point on a circle as the vertex, all angles less than π radians can be represented as an inscribed angle by using two other points on the circle. Since taxicab circles are composed of straight lines, they lack the curvature and "flexibility" of Euclidean circles. This causes some taxicab angles to not be inscribed (the dark line in Figure 4.2.1).

Figure 4.2.1 A taxicab angle that is not an inscribed angle.

From the examples in Figure 4.2.2 it is clear that both the size of an angle and its position affect whether it is inscribed. Some angles just larger than 2 t-radians are not inscribed while other angles just smaller than 4 t-radians are inscribed (for a discussion about t-radians and taxicab angles, see Section 4.1).

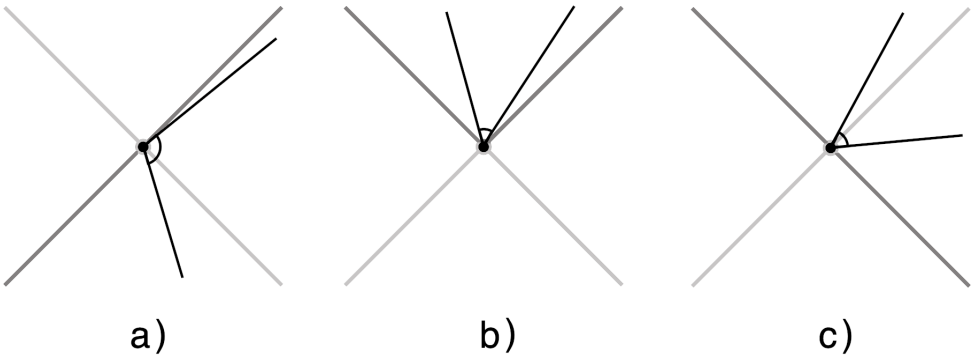


Figure 4.2.2 Inscribed taxicab angles. The angles shown are a) strictly positively inscribed, b) completely inscribed, and c) strictly negatively inscribed.

Since the position of an angle plays a role in whether the angle is inscribed, we introduce here a number of definitions for different types of inscribed angles in taxicab geometry.

Definition 4.2.1 *A taxicab angle is* ***positively inscribed*** *if a line of slope 1 through its vertex remains outside the angle.*

Definition 4.2.2 *A taxicab angle is* ***negatively inscribed*** *if a line of slope -1 through its vertex remains outside the angle.*

Definition 4.2.3 *A taxicab angle is* ***inscribed*** *if it is positively inscribed, negatively inscribed, or both.*

Definition 4.2.4 *A taxicab angle is* ***completely inscribed*** *if it is both positively inscribed and negatively inscribed.*

Definition 4.2.5 *A taxicab angle is* ***strictly positively inscribed*** *if it is positively inscribed but not negatively inscribed. Similarly, a taxicab angle is* ***strictly negatively inscribed*** *if it is negatively inscribed but not positively inscribed.*

Note that the Inscribed Angle Theorem from Euclidean geometry does not hold in taxicab geometry: an angle θ inscribed in a circle is half of the central angle 2θ that intercepts the same arc on the circle. Figure 4.2.3 illustrates two scenarios where the ratio of the inscribed angle to the central angle subtended by the same arc is different. In the left figure, inscribed angle α has measure 1 t-radian and the central angle θ has measure 2 t-radians. In the right figure, inscribed angle α again has measure 1 t-radian but the central angle θ has measure 2.5 t-radians.

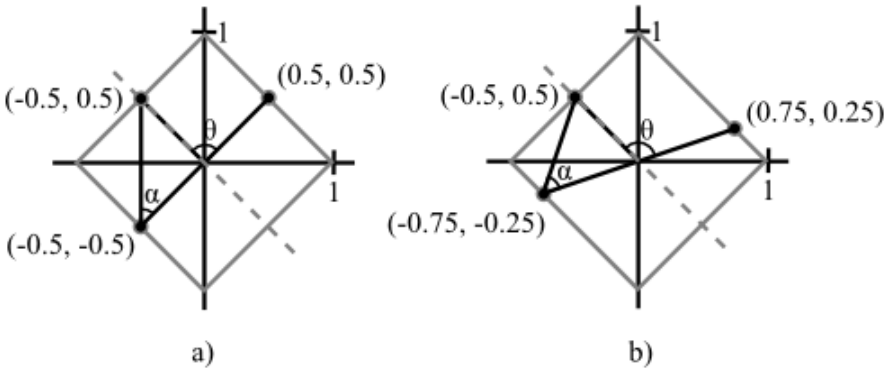


Figure 4.2.3 Failure of the Euclidean Inscribed Angle Theorem in taxicab geometry.

Inscribed angles play a role in the existence of incircles and circumcircles for triangles which are discussed in Chapter 6.

Angles That Are Not Inscribed

It will be helpful to identify a primary characteristic of an angle that is not inscribed. Without loss of generality, anchor one ray of an angle at the origin and place it in the upper half of Quadrant I (the black ray in Figure 4.2.4). The proposed second rays for the angle in light gray will form inscribed angles (using the dotted gray lines we can see that rays 3, 4, 5, or 6 would form a negatively inscribed angle and rays 1, 2, 3, or 4 a positively inscribed angle). The black ray in Quadrant IV illustrates the kind of ray that needs to be drawn to form an angle that is not inscribed. The important characteristic to note is that an angle that is not inscribed will always fully encompass a right angle aligned along lines of slope ± 1, for otherwise one of the dotted lines would remain outside the angle therefore making it inscribed.

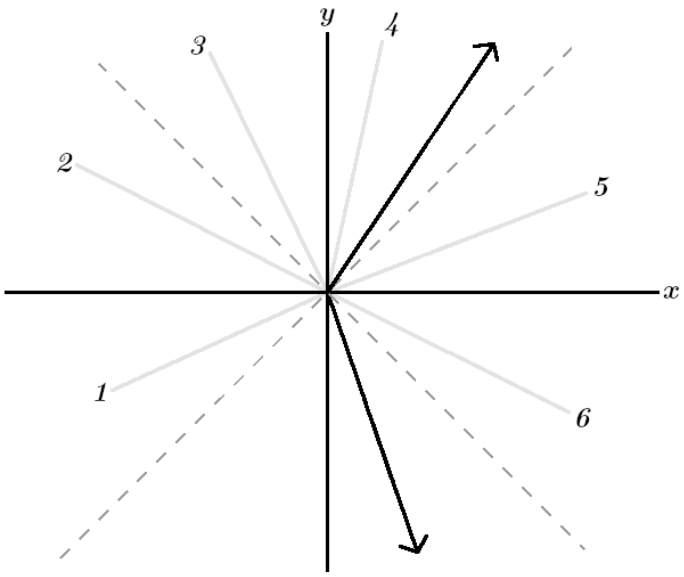


Figure 4.2.4 An angle that is not inscribed (black rays) opens so widely that it will always fully encompass a right angle

Section 4.3: Taxicab Trigonometric Functions

[The reader should note that the earliest research into trigonometry in taxicab geometry by Brisbin and Artola in 1985 used *modified taxicab geometry* (Section 1.3). This is the reason that, on the surface, their results bear no resemblance to what is presented here. Ultimately, their $\cosd\theta$ and $\sind\theta$ functions agree with the cosine and sine functions below, but their definitions rely on Euclidean angles and Euclidean trigonometric functions and are thus not definitions native to *pure taxicab geometry*.]

We now turn to the definition of the trigonometric functions sine and cosine in taxicab geometry. From these definitions familiar formulas for the tangent, secant, cosecant, and cotangent functions can be defined.

Definition 4.3.1 *The point of intersection of the terminal side of a taxicab angle θ in standard position with the taxicab unit circle is the point* $(\cos\theta, \sin\theta)$.

It is important to note that the taxicab sine and cosine values of a taxicab angle do not agree with the Euclidean sine and cosine values of the corresponding Euclidean angle. For example, the angle 1 t-radian has equal taxicab sine and cosine values of 0.5. The range of the cosine and sine functions remains $[-1, 1]$, but the period of these fundamental functions is now 8. It also follows immediately (from the distance function) that $|\sin\theta| + |\cos\theta| = 1$. In addition, the values of cosine and sine vary (piecewise) linearly with θ:

$$\cos\theta = \begin{cases} 1 - \frac{1}{2}\theta, & 0 \le \theta < \pi_t \\ -3 + \frac{1}{2}\theta, & \pi_t \le \theta < 2\pi_t \end{cases} \qquad \sin\theta = \begin{cases} \frac{1}{2}\theta, & 0 \le \theta < \frac{\pi_t}{2} \\ 2 - \frac{1}{2}\theta, & \frac{\pi_t}{2} \le \theta < \frac{3\pi_t}{2} \\ -4 + \frac{1}{2}\theta, & \frac{3\pi_t}{2} \le \theta < 2\pi_t \end{cases} \tag{4.3.1}$$

Table 4.3.1 gives useful straightforward relations readily derived from the graphs of the sine and cosine functions which are shown in Figure 4.3.1. The structure of the graphs of these functions is similar to that of the Euclidean graphs of sine and cosine. But, note that the smooth transition from increasing to decreasing at the extrema has been replaced with a corner. This is the same effect seen when comparing Euclidean circles with taxicab circles.

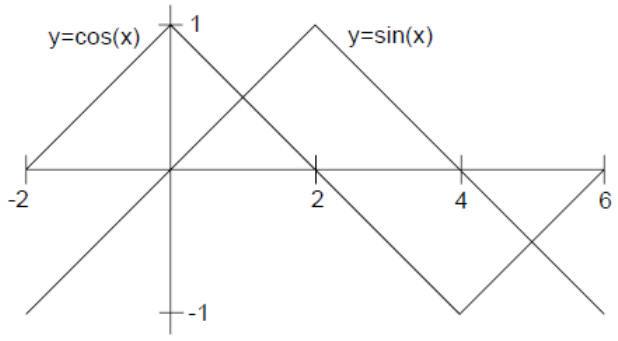


Figure 4.3.1 Graphs of the taxicab sine and cosine functions.

$\sin(-\theta) = -\sin\theta$	$\sin(\theta + \frac{\pi_t}{2}) = \cos\theta$
$\cos(-\theta) = \cos\theta$	$\cos(\theta - \frac{\pi_t}{2}) = \sin\theta$
$\sin(\theta - \pi_t) = -\sin\theta$	$\sin(\theta + 2\pi_t k) = \sin\theta, k \in \mathbb{Z}$
$\cos(\theta - \pi_t) = -\cos\theta$	$\cos(\theta + 2\pi_t k) = \cos\theta, k \in \mathbb{Z}$

Table 4.3.1 Basic Taxicab Trigonometric Relations

With a bit of effort, closed-form expressions can be invented for these piecewise functions. On the interval $[2k-1, 2k)$, the closed-form expressions of sine and cosine are:

$$\cos\theta = \frac{(1+(-1)^k)i^{k-2}(k-1-\frac{1}{2}\theta) + (1+(-1)^{k+1})i^{k-1}(k-\frac{1}{2}\theta)}{2}$$

$$\sin\theta = \frac{(1+(-1)^k)i^{k-2}(k-\frac{1}{2}\theta) + (1+(-1)^{k+1})i^{k-1}(1-k+\frac{1}{2}\theta)}{2}$$

These closed-form expressions are a bit unwieldy to deal with. And, if we look forward to where we are headed, one of the sine and cosine functions switches branches in the piecewise definition every 2 t-radians. So, the expressions we choose to use will be most useful when they are based on 2 t-radian intervals. This will lend itself nicely when we consider functions that are combinations of both sine and cosine such as the tangent function. On the interval $[2k-1, 2k)$, the pseudo closed-form expressions of sine and cosine are:

$$\cos\theta = \begin{cases} i^{k-2}(k-1-\frac{1}{2}\theta), & \text{if } k \text{ is even} \\ i^{k-1}(k-\frac{1}{2}\theta), & \text{if } k \text{ is odd} \end{cases}$$

$$\sin\theta = \begin{cases} i^{k-2}(k-\frac{1}{2}\theta), & \text{if } k \text{ is even} \\ i^{k-1}(1-k+\frac{1}{2}\theta), & \text{if } k \text{ is odd} \end{cases} \tag{4.3.2}$$

where the imaginary number i is used to create positive and negative terms as needed. As an example, choosing $k = 1$ for these gives the expressions

$$\cos\theta = 1 - \frac{\theta}{2} \text{ and } \sin\theta = \frac{\theta}{2}$$

on the interval $[0, 2)$ in agreement with our earlier piecewise definition in Equation 4.3.1.

If we define the taxicab tangent function as in Euclidean geometry as the ratio of sine and cosine, we again have a piecewise-defined function. Since we will be using these in later analysis, the pseudo closed-form expression of the taxicab tangent function is given below. The graph of this function, shown in Figure 4.3.2, has a very similar feel to the Euclidean tangent function with asymptotes at $4k - 2 = \pi_t k - \pi_t/2$ t-radians and a period that is half that of sine and cosine.

$$\tan\theta = \begin{cases} \dfrac{k-\frac{1}{2}\theta}{k-1-\frac{1}{2}\theta}, & \text{if } k \text{ is even} \\ \dfrac{1-k+\frac{1}{2}\theta}{k-\frac{1}{2}\theta}, & \text{if } k \text{ is odd} \end{cases} \tag{4.3.3}$$

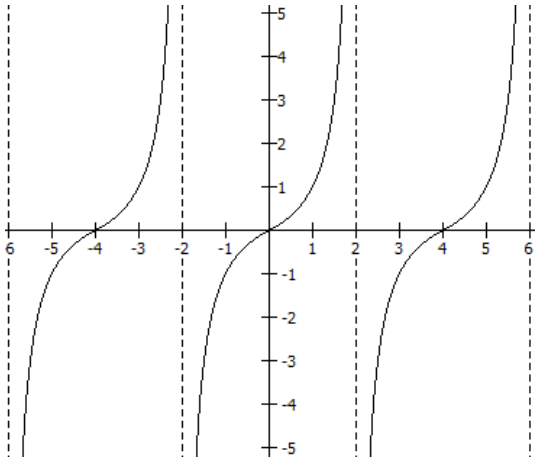


Figure 4.3.2 Graph of the taxicab tangent function.

On the surface, the taxicab tangent function appears to be differentiable everywhere between the asymptotes created where cosine is zero. A formal expression for the derivative would confirm this observation (see Section 8.5).

Since cotangent is simply the reciprocal of tangent, we will leave this to the reader and move to the secant function. Using the reciprocals of the various piecewise branches of the cosine function, we can construct a graph of secant (shown with cosine in Figure 4.3.3).

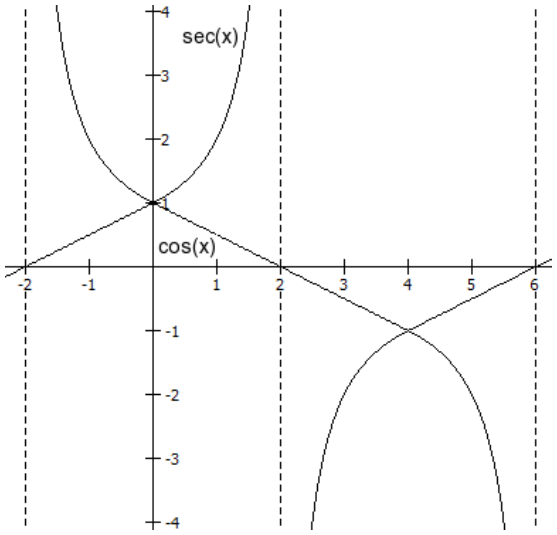


Figure 4.3.3 Graphs of the taxicab secant and cosine functions.

The taxicab secant graph exhibits the same characteristics as the Euclidean secant graph: a period equal to that of cosine; asymptotes where cosine is zero; and, a minimum or maximum where the cosine has a maximum or minimum, respectively. As with tangent, the secant graph is curved. But, unlike the Euclidean secant function, the taxicab secant function is not differentiable at its extrema. This is to be expected since this is where cosine is not differentiable.

As with cotangent, we will leave the details of the cosecant function to the reader.

A Bit of Nostalgia

In Euclidean trigonometry, the mnemonic **SOH-CAH-TOA** is often used to remember the formulae for sine, cosine, and tangent in the context of a right triangle: **S**ine is the length of the **O**pposite side divided by the length of the **H**ypotenuse, **C**osine is **A**djacent over **H**ypotenuse, and **T**angent is **O**pposite over **A**djacent. Unfortunately, this mnemonic does not carry over universally in taxicab geometry. Its application is limited to a particular orientation of the right triangle within the geometry.

In Chapter 6, the concept of a "properly oriented" right triangle is introduced. A right triangle is "properly oriented" if the sides of the triangle forming the right angle are parallel to the coordinate axes (Figure 4.3.4). With this orientation we will show below that the trigonometric functions of the angles can be computed with the mnemonic.

The size of $\angle B$ in Figure 4.3.4 can be found using the Arc Length Theorem (Theorem 4.1.4). A taxicab circle of radius c about point B (gray) will subtend an arc length $s = c \cdot m(\angle B)$. In this case, $s = 2b$ and therefore $m(\angle B) = \frac{2b}{c}$. Using Equation 4.3.1, we have

$$\cos B = 1 - \frac{1}{2}m(\angle B) = 1 - \frac{1}{2}\frac{2b}{c} = \frac{c-b}{c} = \frac{a}{c} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

since $c = a + b$. Similarly,

$$\sin B = \frac{1}{2}m(\angle B) = \frac{1}{2}\frac{2b}{c} = \frac{b}{c} = \frac{\text{opposite}}{\text{hypotenuse}}$$

Repeating the procedure about point A will similarly demonstrate that its trigonometric values can also be found using the standard mnemonic.

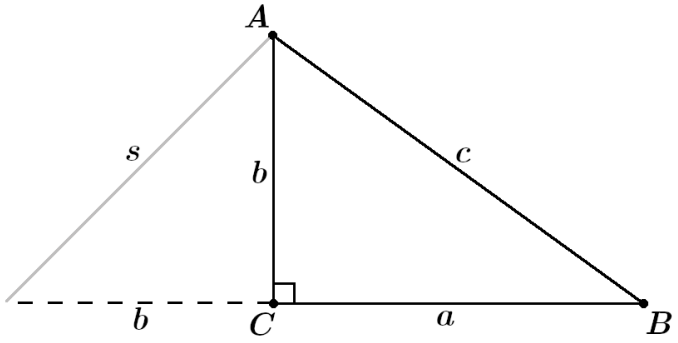


Figure 4.3.4 A "properly oriented" right triangle in taxicab geometry, with a partial taxicab circle of radius c about point B (gray) showing the arc subtended by $\angle B$ on the circle

Section 4.4: Trigonometric Identities

As discussed in Section 1.4, SAS congruence for triangles does not hold in taxicab geometry. Thus, the routine proofs of trigonometric sum and difference formulas are not so routine in this geometry. The first result we will prove is for the cosine of the sum of two angles. The formula given for the cosine of the sum of two angles only takes on two forms; the form used in a given situation depends on the locations of α and β. The notation $\alpha \in I$ will be used to indicate α is an angle in quadrant I and similarly for quadrants II, III, and IV.

Theorem 4.4.1 (Cosine Addition Formula) *For taxicab angles α and β,*

$$\cos(\alpha+\beta) = \pm\,(-1 + |\cos\alpha \pm \cos\beta|) \tag{4.4.1}$$

where the signs are chosen to be negative when α and β are in quadrants on different sides of the x-axis and positive otherwise.

PROOF: Without loss of generality, assume $\alpha, \beta \in [0, 2\pi_t)$, for if an angle θ lies outside $[0, 2\pi_t)$, $\exists k \in \mathbb{Z}$ such that $(\theta + 2\pi_t k) \in [0, 2\pi_t)$ and use of the identity $\cos(\theta + 2\pi_t k) = \cos\theta$ will yield the desired result upon use of the following proof.

All of the sub-cases have a similar structure. We will prove the sub-case $\alpha \in II, \beta \in III$. In this situation, $\frac{3\pi_t}{2} \le \alpha+\beta \le \frac{5\pi_t}{2}$ and so we take the negative signs on the right-hand side of the Equation 4.4.1. Thus,

$$\begin{aligned}
1 - |\cos\alpha - \cos\beta| &= 1 - \left|1 - \frac{1}{2}\alpha - (-3 + \frac{1}{2}\beta)\right| \\
&= 1 - |4 - \frac{1}{2}(\alpha+\beta)| \\
&= \begin{cases} -3 + \frac{1}{2}(\alpha+\beta), & \frac{3\pi_t}{2} \le \alpha+\beta < 2\pi_t \\ 5 - \frac{1}{2}(\alpha+\beta), & 2\pi_t \le \alpha+\beta \le \frac{5\pi_t}{2} \end{cases} \\
&= \cos(\alpha+\beta)
\end{aligned}$$

□

Corollary 4.4.2 (Cosine double angle formula) *For a taxicab angle α,*

$$\cos(2\alpha) = -1 + 2|\cos\alpha| \tag{4.4.2}$$

The curious case structure in Theorem 4.4.1 is due to the odd combinations of quadrants that determine which sign to choose. The reason for the sign change when α and β are on different sides of the x-axis lies in the fact that a corner of the cosine function is being crossed (i.e. different pieces of the cosine function are being used) to obtain the values of the cosine of α and β. Table 4.4.1 summarizes which form of $\cos(\alpha+\beta)$ should be used when.

	α	β
$\cos(\alpha+\beta) = -1 + \lvert\cos\alpha + \cos\beta\rvert$	same	quadrant
	I	II
	III	IV
$\cos(\alpha+\beta) = 1 - \lvert\cos\alpha - \cos\beta\rvert$	I	III
	I	IV
	II	III
	II	IV

Table 4.4.1 Forms of $\cos(\alpha+\beta)$ and Regions of Validity

We can use Theorem 4.4.1 and the relations in Table 4.3.1 to establish a pair of corollaries.

Corollary 4.4.3 (Sine Addition Formula) *For taxicab angles α and β,*

$$\sin(\alpha+\beta) = \pm\,(-1 + \lvert\sin\alpha \pm \cos\beta\rvert) \tag{4.4.3}$$

where the signs are chosen according to Table 4.4.2.

PROOF**:** First, note $\sin\theta = \cos(\theta - \frac{\pi_t}{2})$. As with the cosine addition formula, all cases are proved similarly. We will assume $\alpha \in I$ and $\beta \in IV$. We have $\alpha - \frac{\pi_t}{2}$ and β in the same quadrant, thus

$$\begin{aligned}\sin(\alpha+\beta) &= \cos((\alpha+\beta) - \frac{\pi_t}{2}) \\ &= \cos((\alpha - \frac{\pi_t}{2}) + \beta) \\ &= -1 + \lvert\cos(\alpha - \frac{\pi_t}{2}) + \cos\beta\rvert \\ &= -1 + \lvert\sin\alpha + \cos\beta\rvert\end{aligned}$$

□

	α	β
$\sin(\alpha+\beta) = -1 + \lvert\sin\alpha + \cos\beta\rvert$	I	III
	I	IV
	II	II
	IV	IV
$\sin(\alpha+\beta) = 1 - \lvert\sin\alpha - \cos\beta\rvert$	I	I
	I	II
	II	III
	II	IV
	III	III
	III	IV

Table 4.4.2 Forms of $\sin(\alpha+\beta)$ and Regions of Validity

Corollary 4.4.4 (Sine double angle formula) *For a taxicab angle α,*

$$\sin(2\alpha) = -1 + 2\,|\cos(\alpha - 1)| \tag{4.4.4}$$

PROOF: $\sin(2\alpha) = \cos(2\alpha - \frac{\pi_t}{2}) = \cos(2(\alpha - \frac{\pi_t}{4})) = -1 + 2\,|\cos(\alpha - \frac{\pi_t}{4})|$ □

"Pythagorean" Identities

As noted in Section 4.3, from the taxicab distance function and the definitions of cosine and sine, the taxicab version of the Pythagorean trigonometric identity is

$$|\sin\theta| + |\cos\theta| = 1 \tag{4.4.5}$$

Performing the usual divisions by sine and cosine yield the alternate versions of this identity:

$$|\tan\theta| + 1 = |\sec\theta| \tag{4.4.6}$$

$$1 + |\cot\theta| = |\csc\theta| \tag{4.4.7}$$

Negative Angle Identities

As in Euclidean geometry, the addition formulas for cosine and sine allow for derivation of the negative angle identities stated in Table 4.3.1.

Theorem 4.4.5 (Cosine Negative Angle Formula) *For a taxicab angle α,*

$$\cos(-\alpha) = \cos(\alpha) \tag{4.4.8}$$

PROOF: Using the Cosine Addition Formula in Equation 4.4.1, we have

$$1 = \cos(0) = \cos(\alpha + (-\alpha)) = \pm\,(-1 + |\cos(\alpha) \pm \cos(-\alpha)|)$$

For α in a given quadrant, $-\alpha$ is always in a quadrant that leads to selecting the minus signs based on Table 4.4.1. Therefore, we have

$$1 = 1 + |\cos(\alpha) - \cos(-\alpha)|$$

$$0 = |\cos(\alpha) - \cos(-\alpha)|$$

from which we can conclude that $\cos(-\alpha) = \cos(\alpha)$. □

Theorem 4.4.6 (Sine Negative Angle Formula) *For a taxicab angle* α*,*

$$\sin(-\alpha) = -\sin(\alpha) \tag{4.4.9}$$

PROOF: Since sine is periodic with period $2\pi_t$,

$$\sin(-\alpha) = \sin(2\pi_t + (-\alpha))$$

Case 1: $\alpha \in I$ or $\alpha \in II$. Then either $-\alpha \in IV$ or $-\alpha \in III$ and when employing the Sine Addition Formula in Equation 4.4.3 we select the positive signs based on Table 4.4.2:

$$\begin{aligned}
\sin(-\alpha) &= -1 + |\sin(2\pi_t) + \cos(-\alpha)| \\
&= -1 + |0 + \cos(-\alpha)| \\
&= -1 + |\cos(\alpha)| && \text{(Theorem 4.4.5)} \\
&= -|\sin(\alpha)|
\end{aligned}$$

where we have employed the "Pythagorean" identity in the last step. In the quadrants specified for α the sine function is positive, and therefore we can drop the absolute value and achieve the result $\sin(-\alpha) = -\sin(\alpha)$.

Case 2: $\alpha \in III$ or $\alpha \in IV$. Then either $-\alpha \in II$ or $-\alpha \in I$ and when employing the Sine Addition Formula in Equation 4.4.3 we select the negative signs based on Table 4.4.2:

$$\begin{aligned}
\sin(-\alpha) &= 1 - |\sin(2\pi_t) - \cos(-\alpha)| \\
&= 1 - |0 - \cos(-\alpha)| \\
&= 1 - |-\cos(\alpha)| && \text{(Theorem 4.4.5)} \\
&= 1 - |\cos(\alpha)| \\
&= |\sin(\alpha)|
\end{aligned}$$

where we have employed the "Pythagorean" identity in the last step. In the quadrants specified for α the sine function is negative, and therefore we must change the sign of $\sin(\alpha)$ in order to drop the absolute value, but this achieves the result $\sin(-\alpha) = -\sin(\alpha)$. □

Section 4.5: Polar Coordinate Graphs

In Euclidean geometry some of the most interesting graphs using very simple expressions occur in polar coordinates. Limaçons, lemniscates, roses, and the spiral of Archimedes all have very simple representations in polar coordinates.

At the heart of polar coordinates is the variable r, the distance from the origin. Since taxicab geometry uses a different distance function, the expression for r changes in taxicab geometry. This will in turn affect polar coordinate graphs. Furthermore, since angles are also a kind of distance measure, as we have seen, the various trigonometric functions have changed as well. In this section we will explore polar coordinate graphs in taxicab geometry.

We will investigate these graphs using their Cartesian coordinate equivalents, partly to highlight how the distance function impacts the expressions but also because a taxicab polar coordinate graphing calculator does not exist!

Many of the examples below use trigonometric functions. As explored at the end of Section 4.3, for properly oriented angles taxicab trigonometric functions can be expressed as ratios within a right triangle such as

$$\sin\theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$$

So, we will be able to mimic Euclidean Cartesian coordinate conversions by simply adjusting the expression for r to be taxicab distance.

Limaçons

Limaçons are generated by expressions like $r = a \pm b\sin\theta$ and $r = a \pm b\cos\theta$. If we select one of these, say $r = a + b\sin\theta$, and express it in Cartesian coordinates we have

$$\sqrt{x^2+y^2} = a + b\frac{y}{\sqrt{x^2+y^2}}$$

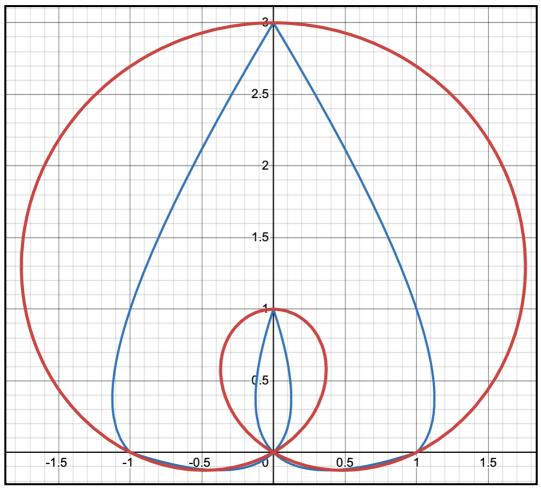
where we have used $\sin\theta = \frac{y}{r}$ on the right-hand side. If we further rearrange to avoid division by zero and include the positive and negative square root we have

$$x^2 + y^2 = \pm a\sqrt{\left(x^2+y^2\right)} + by$$

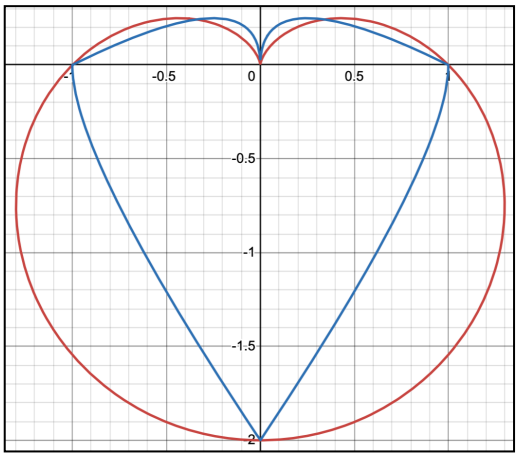
We can see the Euclidean distance function appearing twice here, once explicitly for r itself and again implicitly for r as the denominator of the sine function. If we replace these with the taxicab distance formula we have

$$\left(|x| + |y|\right)^2 = \pm a\left(|x| + |y|\right) + b$$

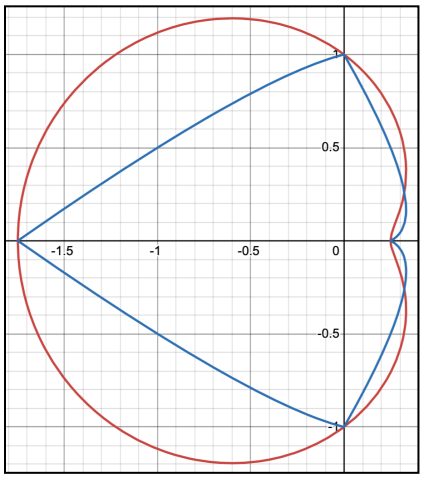
Various taxicab limaçon graphs are shown with their Euclidean counterparts in Figure 4.5.1 using values of $\frac{a}{b}$ that produce markedly different shapes. Notice that the ranges of $\frac{a}{b}$ for loops, dimples, and convexity appear to have the same boundaries as in Euclidean geometry.



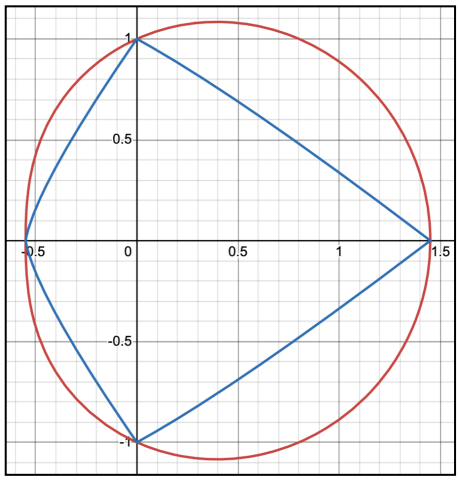
a) Limaçon with inner loop ($\frac{a}{b} < 1$)



b) Cardioid ($\frac{a}{b} = 1$)



c) Dimpled limaçon ($1 < \frac{a}{b} < 2$)



d) Convex limaçon ($\frac{a}{b} \geq 2$)

Figure 4.5.1 Limaçons in Euclidean (red) and taxicab (blue) geometries

A limaçon with $a = b$ is called a cardioid (from Greek "kardia" meaning heart) since it has the general shape of a heart. In this case, taxicab geometry wins the prize because its cardioid is shaped more like a heart than a cardioid in Euclidean geometry!

Lemniscates

A lemniscate is a special case of a Cassini oval which will be studied in Section 4.6 but from a different perspective. Polar equations such as $r^2 = \pm a^2 \cos(2\theta)$ and $r^2 = \pm a^2 \sin(2\theta)$ produce lemniscates which have the shape of a figure '8' or the infinity symbol. The equivalent Euclidean and taxicab Cartesian equations for the positive cosine form are

$$\left(x^2 + y^2\right)^2 + a\left(x^2 + y^2\right) = 2ax^2 \quad \text{and} \quad \left(|x| + |y|\right)^4 + a\left(|x| + |y|\right)^2 = 2ax^2$$

Graphs of a Euclidean and taxicab lemniscate are shown in Figure 4.5.2.

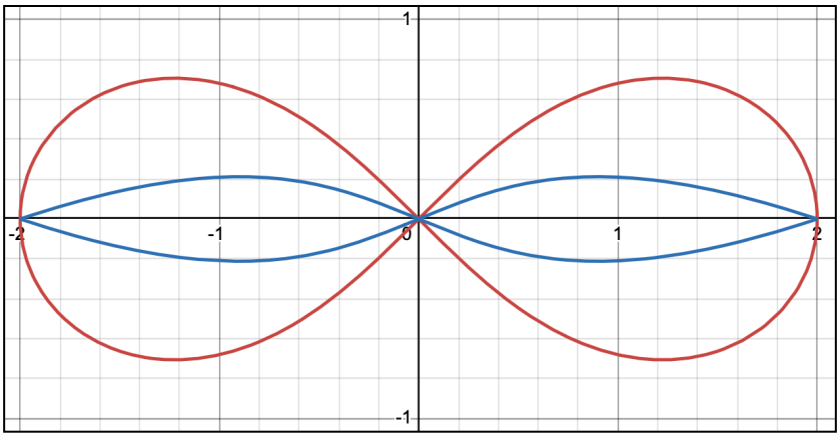


Figure 4.5.2 A lemniscate in Euclidean (red) and taxicab (blue) geometries

Roses

Without a taxicab polar coordinate graphing utility, roses will be more difficult to investigate. But, we can nevertheless produce a few reasonable examples.

A rose is produced from polar equations of the form $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$. If n is odd, the rose has n petals, whereas if n is even there are $2n$ petals. Double-angle trigonometric identities can be employed for $n = 2$ to easily produce Cartesian versions of the polar equations. For instance, the 4-petal rose has a Euclidean form of

$$r = a\sin(2\theta) = 2a\sin\theta\cos\theta$$

which yields

$$\pm(x^2 + y^2)^{3/2} = 2axy$$

where we again have introduced the plus and minus due to the even root. The equation for the corresponding taxicab rose with 4 petals is

$$\pm(|x| + |y|)^3 = 2axy$$

When graphed, the utility just happened to choose green for the color of the taxicab curve which gives the distinct impression that these are not roses but clovers (Figure 4.5.3)!

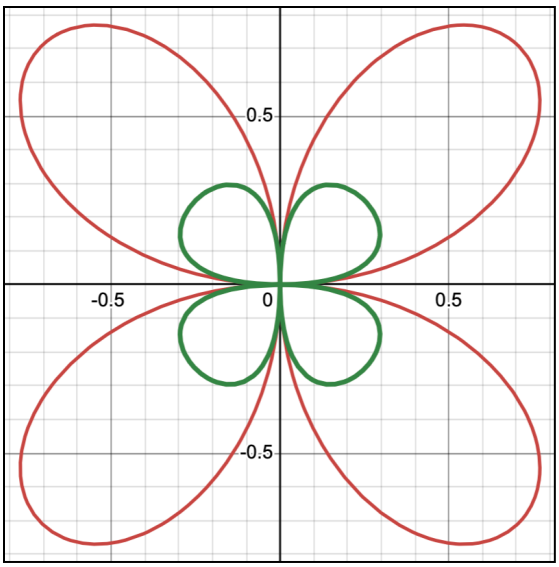


Figure 4.5.3 The Euclidean 4-petal rose (red) becomes a 4-leaf clover (green) in taxicab geometry

For a case where n is odd, consider $r = a\cos(3\theta)$. After a bit of work with additive and double-angle trigonometric identities the equivalent Cartesian equation is

$$(x^2+y^2)^2 = a\left(2x^3 - x(x^2+y^2) - 2xy^2\right)$$

with taxicab equivalent

$$(|x|+|y|)^4 = a\left(2x^3 - x(|x|+|y|)^2 - 2xy^2\right)$$

Figure 4.5.4 shows the result, called "taxicab scissors." Note the loss of rotational symmetry in the taxicab case.

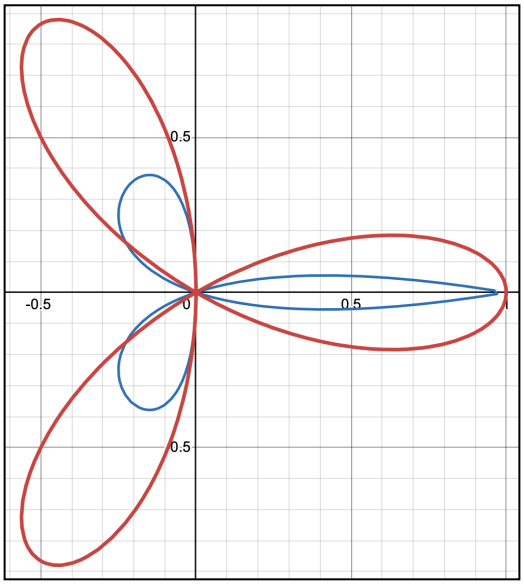


Figure 4.5.4 A Euclidean 3-petal rose (red) and corresponding taxicab scissors (blue)

Spiral of Archimedes

The spiral of Archimedes has a polar equation of $r = \theta$. So, we need a strategy to convert a taxicab angle θ measured in t-radians to Cartesian coordinates. In Quadrant I, an angle θ whose terminal ray intersects a circle of radius r at point (x, y) subtends an arc length of $2y$. By Theorem 4.1.4, the relationship $s = r\theta$ still holds which means

$$\theta = \frac{s}{r} = \frac{2y}{r} = \frac{2y}{|x|+|y|}$$

So, the equation for the spiral in Quadrant I is

$$r = |x|+|y| = \frac{2y}{|x|+|y|} = \theta$$

In Quadrant II the measure of the angle is 2 t-radians plus an acute angle with initial side along the positive y-axis that subtends an arc of length $2|x|$. Since x is negative, this leads to

$$\theta = 2 - \frac{2x}{|x|+|y|} = 2\left(1 - \frac{x}{|x|+|y|}\right)$$

In general, for Quadrant n (with Quadrant V identical to Quadrant I, etc.)

$$\theta = \begin{cases} 2\left((n-1) + \dfrac{(-1)^{\frac{n-1}{2}} y}{|x|+|y|}\right), & n \text{ odd} \\ 2\left((n-1) + \dfrac{(-1)^{\frac{n}{2}} x}{|x|+|y|}\right), & n \text{ even} \end{cases}$$

For example, this leads to the following equations for the spiral in Quadrants II and III:

$$|x|+|y| = 2\left(1 - \frac{x}{|x|+|y|}\right)$$

$$|x|+|y| = 2\left(2 - \frac{y}{|x|+|y|}\right)$$

The result over $[0, 4\pi]$ is shown in Figure 4.5.5.

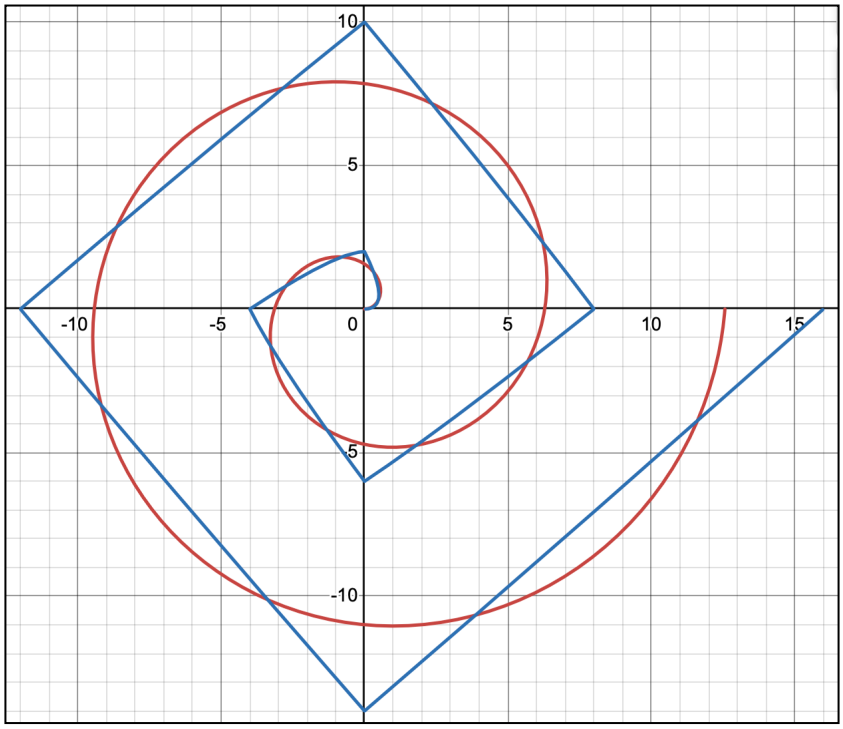


Figure 4.5.5 The Euclidean (red) and taxicab (blue) spiral of Archimedes

None of the taxicab spiral sections are straight lines, but their curvature is decreasing with increasing r in the same manner as the Euclidean spiral so that they quickly begin to have that appearance.

Chapter 5
Geometry

Section 5.1: Regular Polygons

In 2008, Harun Bariş Çolakoğlu and Rüstem Kaya went looking for regular polygons in taxicab geometry. They found ... hardly any! Let's review what regular polygons are and see why there are so few in taxicab geometry.

NOTE: In their paper on regular polygons, Çolakoğlu and Kaya use *modified taxicab geometry*. So, any reference to angles in their paper means Euclidean angles. This is an important distinction because ultimately some of their results will not apply in *pure taxicab geometry*.

Definition 5.1.1 *A **regular polygon** is a polygon (a closed-figure with straight edges) that is both equilateral (the length of all sides are equal) and equiangular (the measure of all interior angles are equal).*

Triangles

In Figure 5.1.1 an equilateral taxicab triangle is presented. But, it is clear that the triangle is neither Euclidean equiangular nor taxicab equiangular. So, in both the pure and modified variants of taxicab geometry, the concept of triangles being equiangular if and only if they are equilateral has been severed.

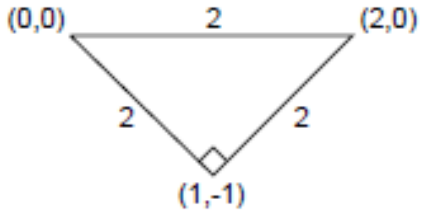


Figure 5.1.1 An equilateral taxicab triangle that is not equiangular.

Corollary 2.1.4 outlines the transformations that preserve taxicab length. If we consider the transformation relationship between two sides of a triangle, we can arrive at the constraints necessary for equilateral taxicab triangles.

Case 1: The slope of two consecutive sides of a triangle are reciprocals. Without loss of generality, let $A = (0, 0)$ and $B = (x, y)$ form the side of a triangle, with $x > y > 0$. Then the sides are reflections in the line $y = x$ and so the third vertex is $C = (y, x)$. The taxicab lengths of $\overline{AB}$ and $\overline{AC}$ are both $x + y$, and the length of $\overline{BC}$ is $2(x - y)$. To have an equilateral triangle, we require $2(x - y) = x + y$ which results in the constraint $x = 3y$. Choosing $c > 0$, we get a triangle like the one pictured in Figure 5.1.2. But, this class of similar equilateral triangles is not equiangular since it can be shown that the measures of the angles are $m(\angle A) = 4/3$ and $m(\angle B) = m(\angle C) = 3/2$.

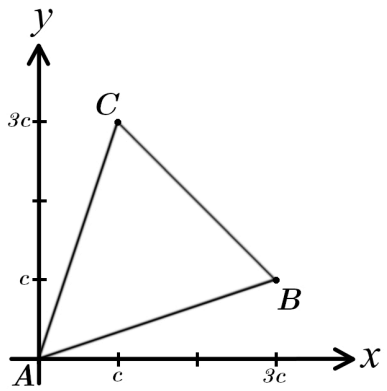


Figure 5.1.2 One of only two kinds of equilateral triangles in taxicab geometry

Case 2: The slope of two consecutive sides of a triangle are negative reciprocals. Without loss of generality, let $A = (0, 0)$ and $B = (x, y)$ form the side of a triangle, with $x, y > 0$. Then the sides are reflections in the line $x = 0$ and so the third vertex is $C = (-x, y)$. The length of $\overline{AB}$ and $\overline{AC}$ are both $x + y$, and the length of $\overline{BC}$ is $2x$. To have an equilateral triangle, we require $2x = x + y$ which results in the constraint $x = y$, i.e. $\overline{AB}$ and $\overline{AC}$ lie along the lines $y = x$ and $y = -x$ with $\overline{BC}$ obviously being horizontal, which is the style of triangle show in Figure 5.1.1. But, as stated earlier, this class of similar equilateral triangles is not equiangular either since $\angle A$ is a right angle.

Case 3: One side of the triangle lies along a coordinate axis. Then the second side must lie along the other axis and the third side along either $y = x$ or $y = -x$. Choosing a side length of c for the horizontal and vertical sides and noting that the length of the third side is then $2c$, it impossible to form an equilateral triangle (since $2c \neq c$ for all $c > 0$).

Therefore, we have the following results.

Theorem 5.1.1 *There are exactly two classes (types) of equilateral triangles in taxicab geometry and neither is equiangular.*

Corollary 5.1.2 *There are no regular taxicab triangles.*

Squares (including taxicab circles!)

Squares are easy to construct and behave much nicer than triangles in taxicab geometry since taxicab length is preserved by right angle rotations (a $y = x$ reflection followed by an $x = 0$ or $y = 0$ reflection). In addition, by Lemma 4.1.3, taxicab right angles are the same as Euclidean right angles, no matter the orientation in the geometry, so taxicab right angles are preserved by any rotation. Therefore we have the following theorem.

Theorem 5.1.3 *All taxicab squares are regular.*

Octagons

Çolakoğlu and Kaya showed that a Euclidean regular octagon is a regular octagon in modified taxicab geometry if one of the octagon's diagonal lines of symmetry is parallel to any one of the lines $x = 0$, $y = 0$, $y = x$, or $y = -x$ (Figure 5.1.3). In this case, each consecutive side is a reflection about a line of symmetry that preserves taxicab length according to Corollary 2.1.4. In addition, the interior *taxicab* angles of an octagon in this orientation are also preserved by the symmetry present, so these particular octagons are also regular in pure taxicab geometry.

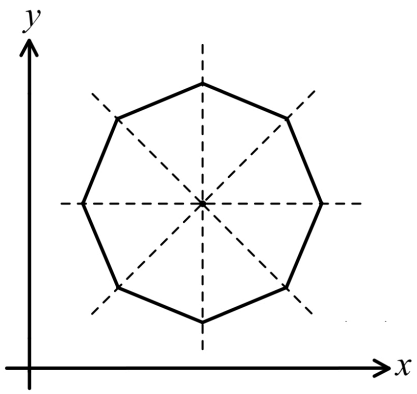


Figure 5.1.3 A regular octagon in taxicab geometry

The researchers then continued to develop a procedure to construct a regular octagon by starting with any given line segment with one such construct shown in Figure 5.1.4. By simply measuring the interior angles along taxicab circles centered at each vertex, it is clear these are not congruent taxicab angles despite being congruent Euclidean angles.

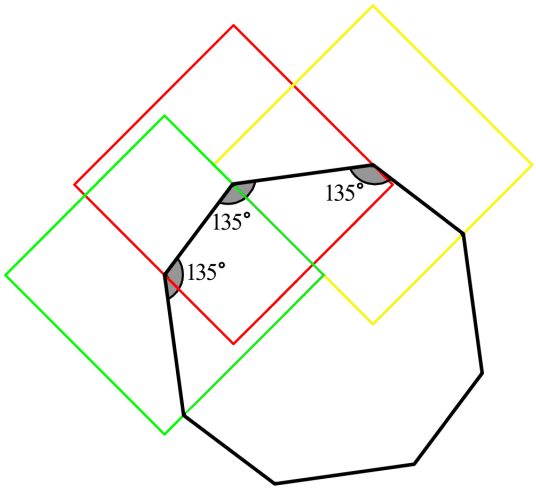


Figure 5.1.4 A octagon that is regular in *modified taxicab geometry* but not in *pure taxicab geometry*

But, could this procedure be modified to produce an polygon with equal *taxicab* angles by rotating an equivalent *taxicab* angle at each vertex? I believe so, it would just have a subtly different shape that the one pictured in Figure 5.1.4.

Theorem 5.1.4 *There exists a regular octagon in pure taxicab geometry having any given line segment as a side.*

Other Polygons

The procedure cited at the end of the Octagons subsection above was applied more generally by Çolakoğlu and Kaya to $2n$-gons. But, no matter the orientation of the starting segment, all hexagons and polygons with more than 8 sides of this nature (with equal *Euclidean* interior angles) will suffer from non-equal interior *taxicab* angles since the set of diagonals of symmetry will extend beyond those that preserve taxicab angle measures. But, as discussed above for octagons, it does appear that the Çolakoğlu-Kaya construction for $2n$-gons can be adapted to rotate a constant *taxicab* angle at each vertex.

Theorem 5.1.5 *There exists a regular $2n$-gon for $n \geq 2$ in pure taxicab geometry having any given line segment as a side.*

While the researchers were unsure about $(2n-1)$-gons in general in *modified taxicab geometry*, it would seem the same would apply for them as well in *pure taxicab geometry*. This leads us to the final proposed conclusions.

Theorem 5.1.6 *The only taxicab regular polygons that are also Euclidean regular polygons are squares and octagons with an axis of symmetry parallel to one of the lines $x = 0$, $y = 0$, $y = x$, or $y = -x$.*

Conjecture 5.1.7 *All regular polygons in pure taxicab geometry have an even number of sides and can be constructed given any segment as a starting side.*

Section 5.2: Angle Sections

In a 2007 research note, Robert J. MacG. Dawson examined some basic axioms in taxicab geometry. One of the axioms that he noted was missing was the ability to bisect a *Euclidean* angle in taxicab geometry. And so we see here again the tendency of researchers to work in *modified taxicab geometry* and choose to change the taxicab metric but not fully recognize it throughout the geometry. But, to Dawson's credit, he does mention in passing that *taxicab* angles can be bisected and even trisected, the latter having been proved impossible in Euclidean geometry by P. L. Wantzel in 1837.

In this section we would like to extend and complete Dawson's work by giving a construction that will not only bisect or trisect a taxicab angle but even *arbitrarily* section it.

Sectioning a Euclidean Angle

Using only a straight edge and a compass, the only angle sections that can be created in Euclidean geometry are from repeated bisection. Given angle $\angle CAB$, the usual bisection construction is as follows (Figure 5.2.1). Use a compass at point A to draw arcs at equal radius intersecting the sides $\overline{AB}$ and $\overline{AC}$ of the angle. At the intersection points of these arcs with the sides of the angle, use the compass to draw arcs of equal radius in the interior of the angle. Connecting the intersection of these arcs with point A will split $\angle CAB$ into two equal angles.

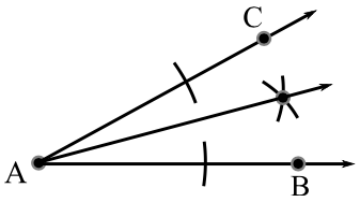


Figure 5.2.1 Construction to bisect a Euclidean angle

Sectioning a Taxicab Angle

As noted by Dawson and others, the t-radian can be sectioned using only a compass and a straight edge into an arbitrary number of equal angles. This is a direct result of the t-radian being defined as the length of a line segment and the fact that a line segment can be split into an arbitrary number of equal parts. With taxicab geometry routinely failing to satisfy theorems taken for granted in Euclidean geometry, such a result is a refreshing change of pace!

While Dawson is correct in his statement, and he provides a taxicab construction to bisect a t-radian angle, it was not immediately obvious whether the arbitrary sectioning of an angle could be done with only *taxicab* constructs. Many Euclidean constructions exist to arbitrarily split a line segment, so Euclidean constructs could be used to arbitrarily section a taxicab angle. But, what would this construction look like with only *taxicab* constructs? Our goal at present is simply to generalize Dawson's construction.

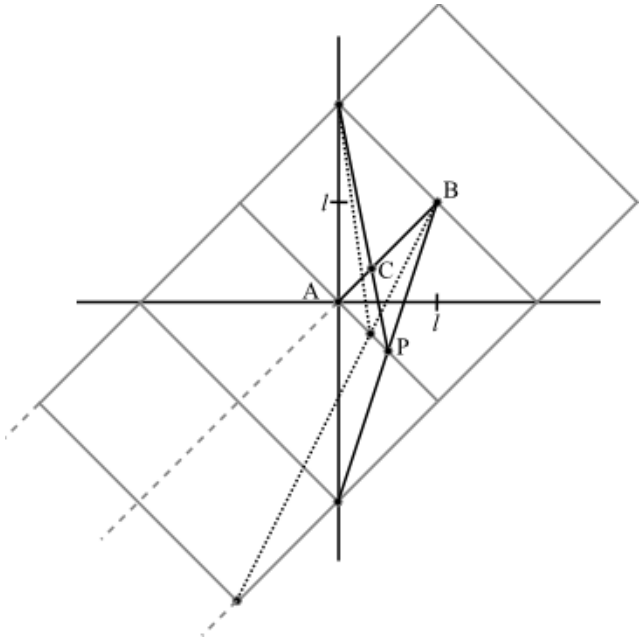


Figure 5.2.2 Construction to split a line segment into n equal parts ($n = 3$ and $n = 4$ shown). Since taxicab angles are simply line segments, this construction is capable of arbitrarily sectioning a taxicab angle.

Construction 5.2.1 *To split a line segment with slope 1 into $n \geq 3$ equal parts using taxicab constructions*

Given a line segment $\overline{AB}$ of taxicab length $2l$ with slope 1 with A at the origin, construct a taxicab circle of radius $2l$ with center at B (Figure 5.2.2). Likewise, construct a taxicab circle of radius $2l$ with center at A. If $n > 3$, at the intersection of the circle around A with the extension of the line segment, construct a circle of radius $2l$. If $n > 4$, repeat the last step $n - 4$ additional times. Construct a line from the bottom corner of the last circle to point B creating an intersection P with the circle about B. Construct a line from P to the top of the circle about A creating an intersection C with the line segment $\overline{AB}$. The taxicab length of segment $\overline{AC}$ is $2l/n$.

PROOF: Let $A = (0, 0)$ and $B = (l, l)$. The bottom corner of the last constructed circle is located at $((3 - n)l, (1 - n)l)$ so the line from such a point to B is

$$y = \frac{n}{n-2}(x - l) + l$$

This line intersects the line $y = -x$ at point P with x-coordinate $x = l/(n - 1)$. The line from this intersection to the point $(0, 2l)$ at the top of the circle centered at A is

$$y = (1 - 2n)x + 2l$$

This line intersects the line $y = x$ at $x = 1/n$ giving a taxicab distance from A to this point C of

$$d(A, C) = \left|\frac{l}{n} - 0\right| + \left|\frac{l}{n} - 0\right| = \frac{2l}{n}$$

□

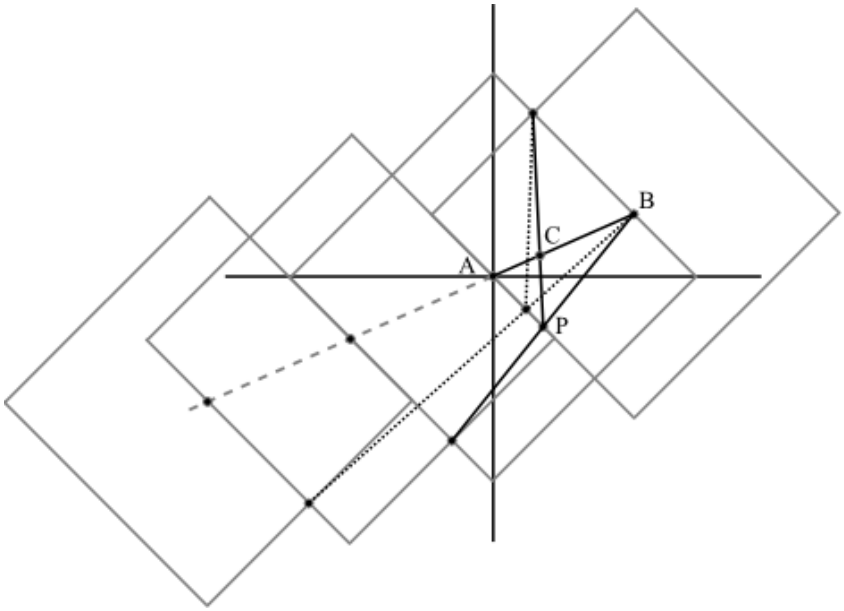


Figure 5.2.3 Generalized construction to split a line segment into n equal parts ($n = 3$ and $n = 4$ shown).

Dawson's construction for bisection is a special case of the above construction with an adjustment because the bottom corner of the last constructed circle is not defined for $n = 2$. This step is merely skipped and the bottom corner of the circle about A is connected to the top of the circle about B which completes the construction.

The presented construction need not be limited to lines of slope 1. This was merely done for the simplicity of calculation and because t-radian angles are defined along diagonal lines. Figure 5.2.3 illustrates the general construction. Not only have we developed a method to arbitrarily section taxicab angles and line segments, but we have also developed yet another method for arbitrarily splitting Euclidean line segments!

Many times there are multiple constructions that accomplish the same result. The challenge now is to find other taxicab constructions to arbitrarily split a line segment into equal parts!

Section 5.3: Parallel Line Theorems

A core foundational scenario studied in Euclidean geometry is the intersection of a pair of parallel lines by a transversal (Figure 5.3.1). In Euclidean geometry, there are a slew of angle relationships formed in this situation. Most of these still hold in taxicab geometry.

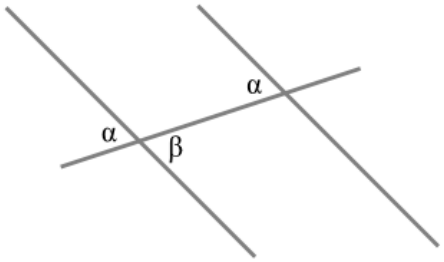


Figure 5.3.1 Parallel lines intersected by a transversal.

Theorem 5.3.1 *Vertical (opposite) angles are congruent.*

PROOF: One angle in a pair of vertical angles can be obtained from the other by a composition of reflections across the x-axis and y-axis (see Figure 5.3.2). Since these transformations preserve taxicab length (Corollary 2.1.4), and taxicab angle measure is equivalent to lengths of line segments, vertical angles are congruent. □

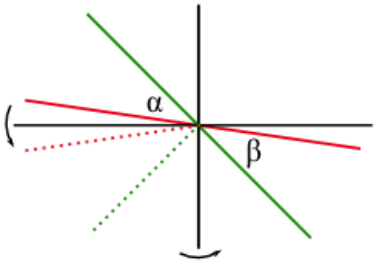


Figure 5.3.2 Obtaining one angle from the other in a pair of vertical angles

Theorem 5.3.2 *Alternate interior angles are congruent.*

PROOF: First, note that the taxicab length of a segment is invariant under translations. So, corresponding angles are clearly congruent. In Figure 5.3.1, angles α and β are vertical angles and thus congruent by Theorem 5.3.1. But, the angle that is alternate interior to β is a translation of α. Therefore, alternate interior angles are congruent. □

Section 5.4: Area of a Square

In Section 3.2 we resolved that the area of a figure in two-dimensional space should agree in Euclidean and taxicab geometry. In particular, for a square this means that in taxicab geometry its area should be $A = s_e^2$ where s_e is the Euclidean length of the sides of the square. To compute this area using purely taxicab lengths, we simply need to adjust the Euclidean length using Theorem 2.1.1.

Theorem 5.4.1 *Let m be the slope of a side s of a taxicab square. Then the area of a square is*

$$A = \frac{(1+m^2)}{(1+|m|)^2} \cdot s^2 \tag{5.4.1}$$

PROOF: If the side of the square is vertical or horizontal, the taxicab and Euclidean lengths will agree and by Theorem 2.1.1 the length correction factor will be 1, and the result follows. If the side of the square is slanted and the slope $m = \frac{a}{b}$ then

$$s_e = \frac{\sqrt{1+m^2}}{1+|m|} \cdot s = \frac{\sqrt{1+(\frac{a}{b})^2}}{1+|\frac{a}{b}|} \cdot s = \frac{\sqrt{a^2+b^2}}{|b|+|a|} \cdot s$$

A side of the square connecting to this side will have slope $m = -\frac{b}{a}$, and its length will be

$$s_e = \frac{\sqrt{1+m^2}}{1+|m|} \cdot s = \frac{\sqrt{1+(-\frac{b}{a})^2}}{1+|-\frac{b}{a}|} \cdot s = \frac{\sqrt{a^2+b^2}}{|b|+|a|} \cdot s$$

So, as expected (since $\pi_t/2$ rotations preserve taxicab length), the length correction factor is identical for all sides of the square and thus

$$A = s_e^2 = \frac{(1+m^2)}{(1+|m|)^2} \cdot s^2$$

□

We could alternatively express the correction factor in terms of the angle of the sides against the horizontal. One of the sides will form an acute angle θ with the horizontal, and its slope will be

$$m = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}\theta}{1-\frac{1}{2}\theta} = \frac{\theta}{2-\theta}$$

Plugging this into Equation 5.4.1 and simplifying gives

$$A = (\frac{1}{2}\theta^2 - \theta + 1) \cdot s^2 \tag{5.4.2}$$

Section 5.5: Parallelograms

As explained in Section 2.5, taxicab geometry does not satisfy the Parallelogram Law. In this section we investigate the relationship between the lengths of the sides and diagonals of a parallelogram in a special case in taxicab geometry.

Theorem 5.5.1 *Given a parallelogram $ABCD$ with taxicab side lengths l and w and diagonal lengths d_1 and d_2 with sides $\overline{AB}$ and $\overline{CD}$ parallel to a coordinate axis and $\angle D$ acute,*

$$d_1 + d_2 = 2l + 2w - 2w\cos D$$

PROOF: Using Figure 5.5.1, $d_1 = (l + c_1) + h = l + w$. Applying Lemma 6.8.1 (a narrow version of the Law of Cosines) on $\triangle ADC$ and noting the side l is parallel to the coordinate axis,

$$d_2 = l + w - 2w\cos D$$

Adding the expressions for d_1 and d_2,

$$d_1 + d_2 = 2l + 2w - 2w\cos D$$ □

If $\angle D$ were a right angle making the parallelogram a rectangle, the Parallelogram Law would reduce to $d_1 + d_2 = 2l + 2w$ which agrees with direct application of the taxicab metric since the taxicab length of each diagonal would be $l + w$.

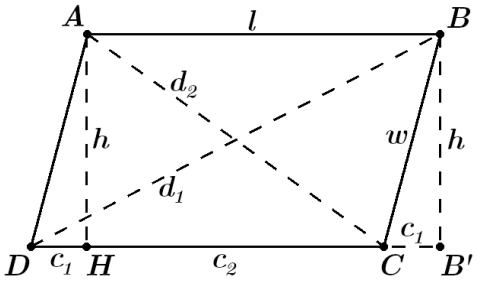


Figure 5.5.1 A parallelogram with sides parallel to the x-axis

If we return to the example parallelogram in Figure 2.5.2, we see that it satisfies the relationship in Theorem 5.5.1: $11 + 9 = 2(6) + 2(5) - 2(5)\frac{1}{5}$.

Chapter 6
Triangles

Section 6.1: Triangle Basics

One fundamental feature of triangles in Euclidean geometry is the sum of the angles is always π radians. A very similar result exists in taxicab geometry. Note that we will be using t-radian measure for taxicab angles as described in Section 4.1.

Theorem 6.1.1 *The sum of the angles of a triangle is $\pi_t = 4$ t-radians.*

PROOF: Given the triangle in Figure 6.1.1, we can translate the angle γ from Q to R and by the congruence of alternate interior angles (Theorem 5.3.2) conclude the sum of the angles of the triangle is 4 t-radians (the size of a straight angle in taxicab geometry). □

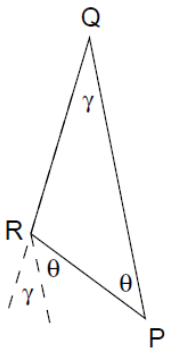
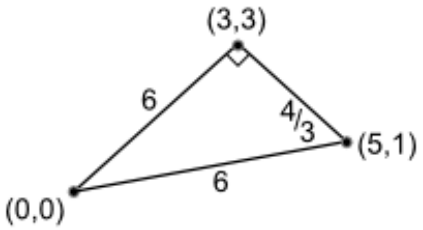


Figure 6.1.1 The sum of the angles of a taxicab triangle is always $\pi_t = 4$ t-radians

Some other very familiar features of Euclidean triangles fail to carry over, however. For example, the triangle shown in Figure 5.1.1 is an equilateral triangle (all sides have the same taxicab length). But, the angles of this triangle are 2 t-radians, 1 t-radian, and 1 t-radian. Therefore, equilateral taxicab triangles are not necessarily equiangular.

In addition, the base angles of an isosceles taxicab triangle need not be congruent as illustrated in Figure 6.1.2. The base angles have measure 2 t-radians (a right angle) and 4/3 t-radians.

Figure 6.1.2 An isosceles taxicab triangle whose base angles are not congruent



Section 6.2: Congruent Triangles

In Euclidean geometry, a good deal of effort can be spent working through congruent triangle relationships, typically a few sections of a textbook. In taxicab geometry it is much simpler. One example kills most kinds of congruent triangle relationships. The triangles shown in Figure 6.2.1 satisfy an ASASA relationship but are not congruent (the right angles are 2 t-radians and the other angles all have measure 1 t-radian). Taking subsets of these letters, we see that SAS, ASA, and AAS all fail to guarantee congruent triangles in taxicab geometry.

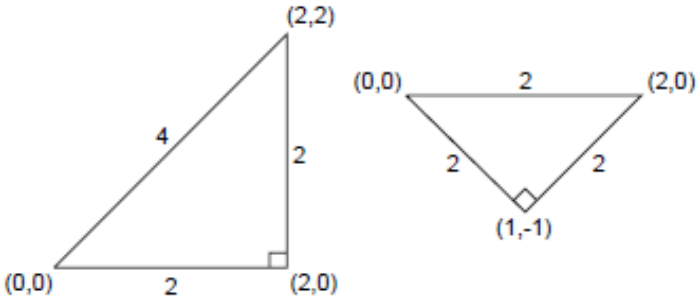


Figure 6.2.1 Triangles satisfying ASASA that are not congruent

In addition, SSS fails based on the triangle in Figure 6.2.2 and the triangle on the right side of Figure 6.2.1.

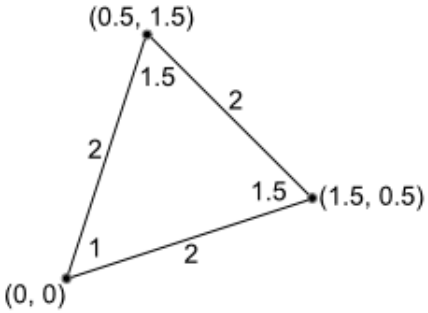


Figure 6.2.2 A triangle that when paired with the triangle on the right side of Figure 6.2.1 fails to satisfy a SSS condition

The only condition that guarantees that two triangles are congruent in taxicab geometry is SASAS. In this case, Theorem 6.1.1 requires the triangles to have an angle sum of $\pi_t = 4$ t-radians, and so the third angles must also be congruent.

Section 6.3: Inscribed Triangles

Triangles in taxicab geometry with one or more inscribed angles (Section 4.2) have special properties. A number of these will be needed in our investigation of taxicab triangle circumcircles and incircles in the following sections.

Theorem 6.3.1 *If a triangle contains a strictly positively inscribed angle, then the other angles are negatively inscribed.*

PROOF: If an angle α of a triangle is strictly positively inscribed, then by way of contradiction, for a neighboring angle to not be negatively inscribed it would have to be larger than $\pi_t - \alpha$ t-radians (see Figure 6.3.1) thus violating the triangle angle requirement (Theorem 6.1.1). □

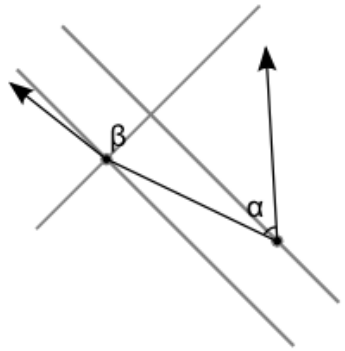


Figure 6.3.1 The neighboring angles of a strictly positively inscribed angle of a triangle must be negatively inscribed. The angle β shown is not negatively inscribed (the line $y = -x$ does not remain outside the angle), but in achieving this, the angle will be too large for a triangle to be formed.

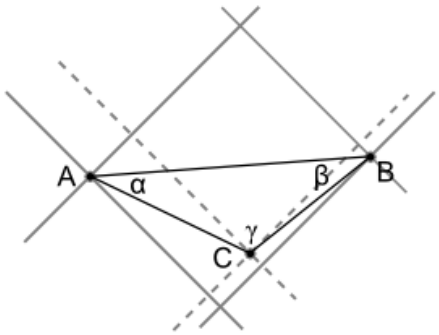
Corollary 6.3.2 *If a triangle contains a strictly negatively inscribed angle, then the other angles are positively inscribed.*

A triangle with a completely inscribed angle can only take on certain shapes and positions. Adding additional completely inscribed angles results in a very small class of triangles as the following theorem illustrates.

Theorem 6.3.3 *An inscribed triangle has three completely inscribed angles if and only if two of its sides have slope -1 or slope 1.*

PROOF: Suppose α and β are completely inscribed angles at vertices A and B of a triangle. Without loss of generality, suppose α is inscribed such that $\overline{AB}$ and $\overline{AC}$ have slope between -1 and 1 with $\overline{AB}$ having a slope strictly less than 1 (Figure 6.3.2). Since β is also completely inscribed, the slope of $\overline{BC}$ is between -1 and 1, inclusive. If the slope of $\overline{BC}$ is strictly less than 1, then there exists a line of slope 1 through C passing through the angle γ making it not positively inscribed. Similarly, if the slope of $\overline{AC}$ is strictly greater than -1, then there exists a line of slope -1 through C passing through the angle γ making it not negatively inscribed. Therefore, the triangle has three completely inscribed angles if and only if the rays of angle γ have slope 1 and -1. □

Figure 6.3.2 Constructing a triangle with three completely inscribed angles. In order for angle γ to be completely inscribed, A and B must be moved to the dotted line intersections so that the rays of γ have slope ± 1. If either A and B is moved farther, the shifted angle will no longer be completely inscribed.



Theorem 6.3.4 *A inscribed triangle with exactly two completely inscribed angles has a side of slope* ± 1.

PROOF: Following on the setup for Theorem 6.3.3 and using Figure 6.3.2, if $\angle C$ is shifted down and to the right to be on the solid line passing through B, the $\angle C$ and the triangle will now be inscribed. But, this means that $\overline{BC}$ has slope 1. □

Inscribed Triangles

Triangles whose angles are all inscribed will be of great interest to us. Having a term for this type of triangle will be useful and, in the end, meaningful. While it may seem presumptuous, calling such a triangle an inscribed triangle is fully justified by the results for triangle circumcircles and incircles.

Definition 6.3.1 *A triangle is an* ***inscribed triangle*** *if all of its angles are inscribed.*

Exploring some properties of inscribed triangles will aid in our study of taxicab triangle circumcircles and incircles. There are minimal properties that an inscribed triangle must have. In fact, by imposing the restriction that all three angles must be inscribed, a more stringent condition results. The following theorem illustrates one of the more fundamental and useful properties of an inscribed triangle.

Theorem 6.3.5 *An inscribed triangle has at least one completely inscribed angle.*

PROOF: By way of contradiction and without loss of generality, assume a triangle has a strictly positively inscribed angle α. By Theorem 6.3.1 the neighboring angles must be negatively inscribed. Suppose one of these angles β is strictly negatively inscribed. Then by Corollary 6.3.2, the other angles must be positively inscribed. So, γ is completely inscribed (Figure 6.3.3). Therefore, an inscribed triangle cannot be constructed with no completely inscribed angles. □

Definition 6.3.2 *A triangle is* ***minimally inscribed*** *if it has exactly one completely inscribed angle.*

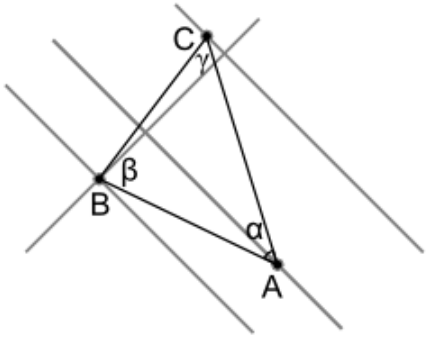


Figure 6.3.3 An inscribed triangle must have at least one completely inscribed angle (angle γ)

The following theorem is an extension and generalization of Theorem 6.3.1 where a hint of alternating inscribed angle properties among the angles of a triangle was seen. Theorem 6.3.5 above provides the flexibility in obtaining the following result.

Theorem 6.3.6 N*eighboring angles of an inscribed triangle have opposite inscribed angle properties (completely inscribed angles may be selected to be either positively or negatively inscribed as needed).*

PROOF: If an angle of a triangle is strictly positively inscribed, then by Theorem 6.3.1 the other angles are negatively inscribed. By Theorem 6.3.5 one of these angles must be completely inscribed. Therefore, neighboring angles in this triangle have opposite inscribed angle properties. The result is similar when beginning with a strictly negatively inscribed angle. The only remaining case is a triangle with only completely inscribed angles. The result follows immediately. □

This alternating property in an inscribed triangle is key to the construction of taxicab circumcircles for triangles.

Section 6.4: Inscribed Circles

In Euclidean geometry, every triangle has a unique inscribed circle. Taxicab geometry, in the usual spirit of a red-headed step-child, does not offer such a broad guarantee.

Regarding inscribed circles, the foundational difference between Euclidean geometry and taxicab geometry is the nature of inscribed angles. Since not all angles are inscribed, there exist triangles that are not inscribed triangles. We will see that this causes not all taxicab triangles to have inscribed circles.

The traditional definition of a triangle incircle is the largest circle inscribed within the triangle that is tangent to all three sides of the triangle. Such a circle is guaranteed to exist in Euclidean geometry. As with circumcircles, a taxicab triangle incircle only exists under certain conditions. We begin by formalizing the definition of an incircle in taxicab geometry to clear up concepts like "tangent" which may not transfer precisely.

Definition 6.4.1 *An **inscribed circle** (incircle) of a taxicab triangle is a taxicab circle entirely contained in a triangle with three of its corners touching the sides of the triangle.*

Theorem 6.4.1 (Triangle Incircle Theorem) *A triangle has a unique taxicab incircle if and only if it is an inscribed triangle.*

PROOF: Let $\triangle ABC$ be an inscribed triangle. By Theorem 6.3.5, the triangle has at least one completely inscribed angle. Without loss of generality, let γ be an inscribed angle in the triangle at vertex C such that $\overline{AC}$ and $\overline{BC}$ have slope between -1 and 1, inclusive (Figure 6.4.1). If the triangle has three inscribed angles, choose γ to be the angle at the intersection of the lines with slope 1 and -1 (see Theorem 6.3.3). For the side $\overline{AB}$ of the triangle opposite γ, consider a point P with distance from B

$$r_\beta = \frac{\alpha \cdot \overline{AB}}{\alpha + \beta}$$

The distance from A to P is

$$r_\alpha = \overline{AB} - r_\beta = \frac{\beta \cdot \overline{AB}}{\alpha + \beta}$$

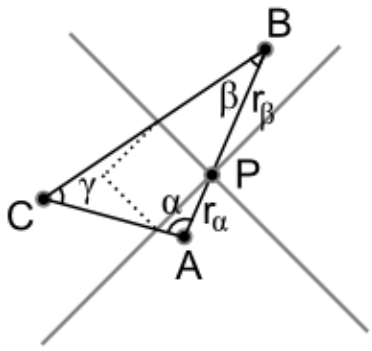


Figure 6.4.1 Construction of the taxicab incircle of an inscribed triangle

The arcs of taxicab circles of radius r_α and r_β centered on A and B, respectively, intersect $\overline{AB}$ at the same point P and by Theorem 4.1.4 both have a length across the interior of the triangle of

$$l = r_\alpha \cdot \alpha = r_\beta \cdot \beta = \frac{\alpha \cdot \beta \cdot \overline{AB}}{\alpha + \beta}$$

Therefore, these arcs form half a taxicab circle inside the triangle. The other half of the taxicab circle remains inside the triangle because the slope of the lines forming the circle are by assumption greater than the slopes of the lines forming the side of the triangle they intersect.

To prove the converse, assume a triangle has a taxicab incircle. If one or zero sides of the circle overlap the triangle (as in Figure 6.4.1), both sides of each angle of the triangle are intersected by a line of slope -1 or 1. This implies a line of the same slope passing through the vertex will remain outside the angle making each angle inscribed. If two sides of the circle overlap the triangle, the enclosed angle is by definition completely inscribed. The other two angles are inscribed by the argument above. Therefore, the triangle is an inscribed triangle. □

For a triangle that is not inscribed, an inscribed taxicab circle still exists in a sense. The enclosed circle technically fails to be an incircle by our definition because it does not touch some of the sides of the triangle (see Figure 6.4.2). The problem centers on the fact that the triangle is significantly "wider" than it is "tall" and taxicab circles are in a sense not as "flexible" as Euclidean circles.

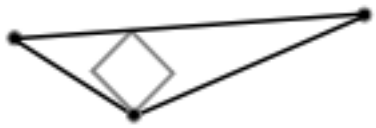


Figure 6.4.2 A taxicab circle within a triangle that is not inscribed

Section 6.5: Circumscribed Circles

In Euclidean geometry, every triangle has a unique circumscribed circle. In taxicab geometry there are limitations on which triangles have circumcircles.

Regarding circumscribed circles, the foundational difference between Euclidean geometry and taxicab geometry is the nature of inscribed angles. Since not all angles are inscribed, there exist triangles that are not inscribed triangles. We will see that this causes not all taxicab triangles to have circumscribed circles.

The traditional definition of a triangle circumcircle, or circumscribed circle, is a circle that passes through the vertex points of the triangle. In Euclidean geometry, all triangles have a unique circumcircle. In taxicab geometry, the existence of circumcircles is more restrictive.

Theorem 6.5.1 (Triangle Circumcircle Theorem) *A triangle has a taxicab circumcircle if and only if it is an inscribed triangle.*

PROOF: If a taxicab circumcircle passes through the vertices of a triangle and completely encloses the triangle, by definition the angles of the triangle are inscribed since a taxicab circle is composed of lines of slope 1 and -1.

To prove the converse, suppose the triangle is an inscribed triangle. By Theorem 6.3.6, neighboring angles of the triangle have opposite inscribed angle properties. Therefore, we may select three lines of slope 1 or -1 through the vertices of the triangle in an alternating fashion to construct three sides of a taxicab circle around the triangle. Since at least one angle of the triangle is completely inscribed, this can be done in multiple ways. At the completely inscribed angle, a line should be selected that maximizes the side of the circle enclosed by the 3 lines (the solid lines of Figure 6.5.1, with the dotted lines showing the alternate choice). By construction, the fourth side of the taxicab circle will enclose the triangle. □

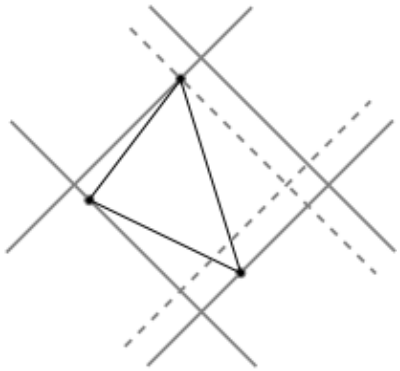


Figure 6.5.1 Construction of the taxicab circumcircle of an inscribed triangle.

The Triangle Circumcircle Theorem justifies our name for inscribed triangles. This type of triangle is precisely the kind for which circumscribed circles exist. From this result, we can easily obtain the taxicab version of the Euclidean geometry "three points define a unique circle" theorem.

Corollary 6.5.2 (Three-point Circle Theorem) *Three non-collinear points lie on a taxicab circle if and only if the triangle they form is inscribed.*

So, in contrast to Euclidean geometry, three non-collinear points do not always lie on a circle in taxicab geometry. If the triangle formed by the points contains an angle that is not inscribed, then no taxicab circle can be drawn passing through the points.

It should also be carefully noted that taxicab triangle circumcircles are not necessarily unique. (This means three non-collinear points do not necessarily lie on a *unique* taxicab circle.) All non-unique cases occur when one or more sides of the circumcircle pass through two vertices. This often creates flexibility in defining possible circumcircles. Three cases are shown in Figure 6.5.2. In the first example, exactly one side has a slope of 1 or -1 and the other sides are of equal taxicab length and longer than the first side (noting that the triangle does not necessarily need to be acute). In this case a circumcircle can be shifted diagonally within a limited range around the triangle. For the second example, one side has slope 1 or -1 and another side is parallel to an axis. Larger and larger circles encompass this type of triangle. These circles also encompass the third example along with another set of circles in a limited range of movement as in the first example. In this case the triangle is isosceles (but not equilateral) with two sides having slope 1 or -1.

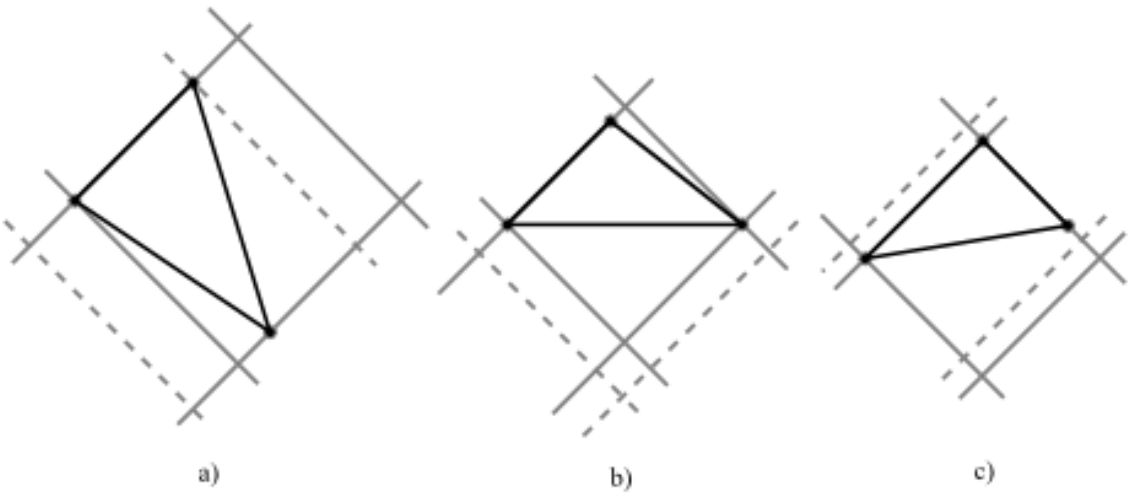


Figure 6.5.2 Triangles with infinite circumcircles

Section 6.6: Area of a Triangle

In Section 3.2 we resolved that the area of a figure in two-dimensional space should agree in Euclidean and taxicab geometry. In particular, for a triangle this means that in taxicab geometry its area should be $A = \frac{1}{2} b_e h_e$ where b_e and h_e are the Euclidean lengths of the base and height of the triangle, respectively. To compute this area using purely taxicab lengths, we simply need to adjust the two Euclidean lengths using Theorem 2.1.1.

Theorem 6.6.1 *Let m be the slope of the base b of a taxicab triangle with corresponding height h. Then the area of a triangle is*

$$A = \frac{(1+m^2)}{2(1+|m|)^2} \cdot b \cdot h \tag{6.6.1}$$

PROOF: If the base is horizontal or vertical, then the by Corollary 2.1.2 the taxicab lengths of the base and height are the same as the Euclidean lengths. This implies the area of the triangle should be $A = bh$ and this is indeed the result we get here since the correction factor is equal to 1 when either $m = 0$ or $m \to \infty$.

Now assume the base is neither horizontal nor vertical. Since the base and height of a triangle are perpendicular, $m_h = -1/m$. Thus, by Theorem 2.1.1 the base has Euclidean length

$$b_e = \frac{\sqrt{1+m^2}}{1+|m|} \cdot b$$

and the height has Euclidean length

$$\begin{aligned} h_e &= \frac{\sqrt{1+m_h^2}}{1+|m_h|} \cdot h \\ &= \frac{\sqrt{1+1/m^2}}{1+|1/m|} \cdot h \\ &= \frac{\sqrt{1+m^2}}{1+|m|} \cdot h \end{aligned}$$

(It should not be surprising that the correction factor for the base and height are the same. Since right angle rotations preserve taxicab length, two perpendicular line segments would naturally have the same correction factor with respect to Euclidean length.)

Solving the above for b_e and h_e and substituting into $A = b_e h_e/2$ gives

$$A = \frac{(1+m^2)}{2(1+|m|)^2} \cdot b \cdot h$$ □

As we had seen with length of line segments, the orientation of a figure affects the calculation of its characteristics.

Rüstem Kaya additionally noticed that this result can easily be extended to (Euclidean) regular polygons. The area of a (Euclidean) regular polygon is the product of the number of sides, the length of one of the sides, and the length of the apothem (a line extending from the center of the polygon to the midpoint of one of the sides, which is perpendicular to the side). Similarly correcting the two lengths using the Euclidean-taxicab length factor, we have

$$A = \frac{n(1+m^2)}{2(1+|m|)^2} \cdot b \cdot a$$

where b is the taxicab length of the side and a is the taxicab length of the apothem. (Hidden in this is the fact that the length conversion factors from Euclidean to taxicab geometry for a and b are equal since such factors are the same for perpendicular lines, something that is highlighted in the proof of Theorem 5.4.1.)

Heron's Formula

If none of the heights of a triangle are known, the usual area formula employed above cannot be used. In this case, Heron's formula is used since it only involves the lengths of the sides of the triangle. For a Euclidean triangle with sides of length a_e, b_e, and c_e,

$$A = \sqrt{p(p-a_e)(p-b_e)(p-c_e)}$$

where $p = (a_e + b_e + c_e)/2$ is the semi-perimeter (half the perimeter) of the triangle. Unfortunately, in taxicab geometry Heron's area formula will in most cases depend on other coordinate differences beyond just the lengths of the sides. In 2003, Münevver Özcan and Rüstem Kaya derived the taxicab version of Heron's formula which is presented here with minor changes and improvements.

Lemma 6.6.2 *For $\triangle ABC$ with side $\overline{BC}$ parallel to a coordinate axis and $\angle B$ and $\angle C$ acute, the area of the triangle is*

$$A = \frac{1}{2}a(p-a)$$

PROOF: Drop an altitude of length h from A to a point A' on side $\overline{BC}$ (Figure 6.6.1). Using $a' = c - h$ we have

$$\begin{aligned} h &= b - (a - a') \\ &= b - a + (c - h) \end{aligned}$$

so that $h = \frac{1}{2}(b + c - a)$. Substituting into the area formula gives

$$\begin{aligned} A = \frac{1}{2}a \cdot h &= \frac{1}{4}a(b + c - a) \\ &= \frac{1}{4}a(a + b + c - a - a) \\ &= \frac{1}{4}a(2p - 2a) \\ &= \frac{1}{2}a(p - a) \end{aligned}$$

□

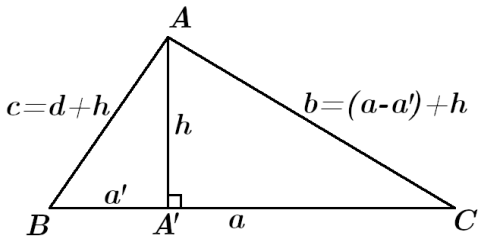


Figure 6.6.1 Computing the area of a triangle with one side parallel to an axis and the angles associated with it acute

Lemma 6.6.3 *For* $\triangle ABC$ *formed by points* $A = (x_a, y_a)$, $B = (x_b, y_b)$, *and* $C = (x_c, y_c)$ *with side* $\overline{BC}$ *parallel to a coordinate axis and* $\angle C$ *an obtuse angle, then the area of the triangle is*

$$A = \frac{1}{2}a\left(p - \max\{\,|x_b - x_c|, |y_b - y_c|\,\}\right)$$

PROOF: Let $\overline{BC}$ be parallel to the x-axis. Drop an altitude of length h from A to a point A' on the extension of $\overline{BC}$ (Figure 6.6.2). Form two expressions for h using the two right triangles:

$$h = c - (a + a')$$

$$h = b - a'$$

Setting these equal and simplifying gives $c = a + b$. Therefore,

$$\begin{aligned} A = \frac{1}{2}a \cdot h &= \frac{1}{2}a\left(c - (a + a')\right) \\ &= \frac{1}{2}a\left(\frac{c + c}{2} - |x_b - x_c|\right) \\ &= \frac{1}{2}a\left(\frac{a + b + c}{2} - |x_b - x_c|\right) \\ &= \frac{1}{2}a\left(p - |x_b - x_c|\right) \end{aligned}$$

If $\overline{BC}$ is parallel to the y-axis, a similar derivation will arrive at $A = \frac{1}{2}a\left(p - |y_b - y_a|\right)$. The max function properly selects between these two orientations since the opposite coordinate difference will always be zero. □

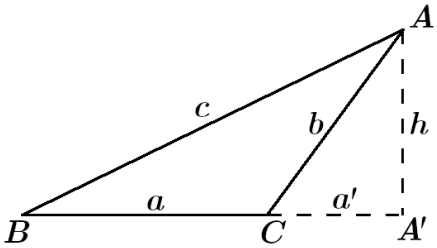


Figure 6.6.2 Computing the area of a triangle with one side parallel to an axis and one angle associated with it obtuse

The primary change in the approach to proving Heron's formula from Özcan and Kaya is the avoidance of the "base vertex," "base line," and "base segment" terminology. These concepts were simply separating the case where the vertex angle split by a line parallel to an axis encompasses a full and "properly oriented" right angle or not.

Theorem 6.6.4 (Heron's Formula) *Let* $\triangle ABC$ *be a triangle formed by points* $A = (x_a, y_a)$, $B = (x_b, y_b)$, *and* $C = (x_c, y_c)$ *where a line* l_1 *parallel to a coordinate axis through vertex* C *intersects the side* $\overline{AB}$ *of the triangle at point* $D = (x_d, y_d)$. *Then, the area of the triangle is*

$$A = \begin{cases} \frac{1}{2}|x_c - x_d|\left(p - \max\{|x_a - x_b|, |x_b - x_c|, |x_a - x_c|\}\right), & \angle C \text{ opens horizontally} \\ \frac{1}{2}|y_c - y_d|\left(p - \max\{|y_a - y_b|, |y_b - y_c|, |y_a - y_c|\}\right), & \angle C \text{ opens vertically} \end{cases} \tag{6.6.2}$$

where $p = \frac{1}{2}(a + b + c)$ *is the semi-perimeter and*

$$x_d = x_a + \frac{(x_b - x_a)(y_c - y_a)}{y_b - y_a} \quad \textit{and} \quad y_d = y_a + \frac{(y_b - y_a)(x_c - x_a)}{x_b - x_a}$$

PROOF: Let l_2 be a line through the vertex of $\angle C$ and perpendicular to l_1 (so parallel to the opposite coordinate axis as l_1).

Case 1: l_2 does not intersect the side $\overline{AB}$ (Figure 6.6.3). In this case, the top triangle (Triangle #1) satisfies the conditions of Lemma 6.6.2 and the bottom triangle (Triangle #2) satisfies the conditions of Lemma 6.6.3. Summing the areas of these triangles gives

$$\begin{aligned} A = A_1 + A_2 &= \frac{1}{2}d(p_1 - d) + \frac{1}{2}d(p_2 - (d + d')) \\ &= \frac{1}{2}d\left(p_1 + p_2 - d - (d + d')\right) \end{aligned}$$

The semi-perimeters of the triangles are $p_1 = \frac{1}{2}(b + c_1 + d)$ and $p_2 = \frac{1}{2}(d + c_2 + a)$, which when added together gives

$$\begin{aligned} p_1 + p_2 &= \frac{1}{2}(b + c_1 + d + d + c_2 + a) \\ &= \frac{1}{2}(a + b + c + 2d) \\ &= p + d \end{aligned}$$

Therefore, with $d = |x_c - x_d|$ and $d + d' = |x_b - x_c|$, the area simplifies to the desired result:

$$\begin{aligned} A &= \frac{1}{2}d(p + d - d - (d + d') \\ &= \frac{1}{2}|x_c - x_d|\left(p - |x_b - x_c|\right) \end{aligned}$$

If the labeling of A and B is interchanged, the term $|x_b - x_c|$ will become $|x_a - x_c|$. In either case, the max function is selecting the largest of the x-coordinate differences among the vertices of the triangle.

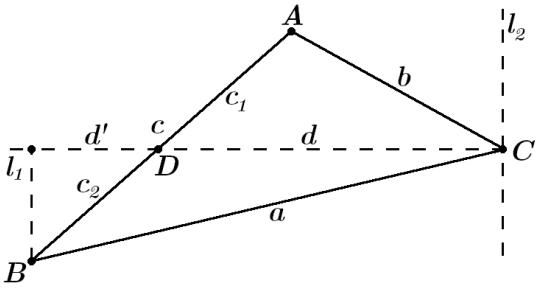


Figure 6.6.3 Computing the area of a triangle where a vertex angle has only one line parallel to the coordinate axes intersecting the opposite side of the triangle

In addition, if the labelling of the axes is interchanged, $\angle C$ will open vertically and the same derivation above will produce and area formula of $A = \frac{1}{2}|y_c - y_d|\left(p - |y_b - y_c|\right)$, or if the labeling of A and B is also interchanged, $A = \frac{1}{2}|y_c - y_d|\left(p - |y_a - y_c|\right)$. Again the max function is properly selecting the largest of the y-coordinate differences among the vertices of the triangle.

Case 2: l_2 intersects the side $\overline{AB}$ (Figure 6.6.4). In this case, since $\angle C$ contains a full, properly oriented right angle, the angle created by the horizontal line and side $\overline{AC}$ is obtuse. So, both the top triangle (Triangle #1) and the bottom triangle (Triangle #2) satisfy the conditions of Lemma 6.6.3. Summing the areas of these triangles gives

$$\begin{aligned} A = A_1 + A_2 &= \frac{1}{2}d_2\left(p_1 - (d_2 + d_3)\right) + \frac{1}{2}d_2(p_2 - \left(d_1 + d_2\right)) \\ &= \frac{1}{2}d_2\left(p_1 + p_2 - d_2 - (d_1 + d_2 + d_3)\right) \end{aligned}$$

The semi-perimeters of the triangles are $p_1 = \frac{1}{2}(b + c_1 + d_2)$ and $p_2 = \frac{1}{2}(a + c_2 + d_2)$, which when added together gives

$$\begin{aligned} p_1 + p_2 &= \frac{1}{2}(a + b + c + 2d_2) \\ &= \frac{1}{2}(2p + 2d_2) \\ &= p + d_2 \end{aligned}$$

Therefore, with $d_2 = |x_c - x_d|$ and $d_1 + d_2 + d_3 = |x_b - x_a|$, the area simplifies to

$$\begin{aligned} A &= \frac{1}{2}d_2\left(p + d_2 - d_2 - (d_1 + d_2 + d_3)\right) \\ &= \frac{1}{2}|x_c - x_d|\left(p - |x_b - x_a|\right) \end{aligned}$$

As with Case 1, the term $|x_b - x_a|$ is selecting the largest of the x-coordinate differences among the vertices of the triangle as can be seen by relabeling the points A and B. And, a relabeling of the coordinate axes similarly switches the formula to y-coordinate differences. Therefore, the result holds.

The expression for x_d can be found by intersecting the line passing through A and B with the line $y = y_c$ or $x = x_c$ depending on whether $\angle C$ is viewed as opening horizontally or vertically. □

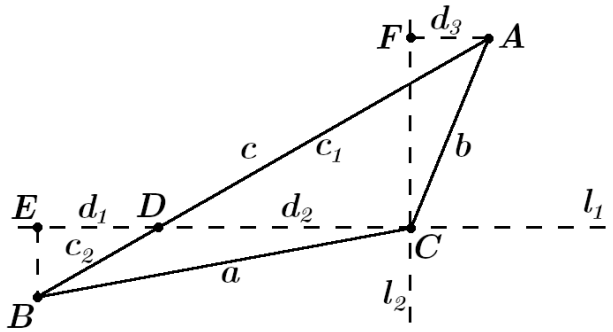


Figure 6.6.4 Computing the area of a triangle where a vertex angle has two lines parallel to the coordinate axes intersecting the opposite side of the triangle

Note that Lemma 6.6.2 and Lemma 6.6.3 are covered in Theorem 6.6.4 in which point D coincides with point B.

As mentioned, Heron's formula in taxicab geometry depends on more than just the lengths of the sides. But, this formulation only adds dependencies on differences of various coordinates of the points forming the triangle.

Section 6.7: Taxicab Pythagorean Theorem

By the very nature of the taxicab distance formula, for a right triangle with legs of length a and b that are parallel to the coordinate axes, the length of the hypotenuse is $c = a + b$. This is in essence the taxicab equivalent of the Euclidean Pythagorean Theorem ($c^2 = a^2 + b^2$). And, it would be easy to leave things here and move on to other topics. However, as discussed in Section 2.1, the orientation of a line segment affects its taxicab length. Therefore, if we rotate a Euclidean triangle, the Euclidean Pythagorean Theorem will still hold but the taxicab length of c will no longer be $a + b$ (see Figure 6.2.1 for an example). In this section, we will explore a full replacement for the Pythagorean Theorem in taxicab geometry.

Definition 6.7.1 *A right angle is* ***properly oriented*** *if its rays are parallel to the coordinate axes.*

Definition 6.7.2 *A right angle* ***opens horizontally*** *a horizontal line passing through the vertex remains inside the angle.*

Definition 6.7.3 *A right angle* ***opens vertically*** *a vertical line passing through the vertex remains inside the angle.*

Theorem 6.7.1 (Taxicab Pythagorean Theorem) *For a right triangle formed by the points $A = (x_a, y_a)$, $B = (x_b, y_b)$, and $C = (x_c, y_c)$ with $\angle C$ being a right angle,*

$$c = \begin{cases} a + b, & \text{if } \angle C \text{ is properly oriented} \\ a + b - 2\min\{\,|x_a - x_c|\,, |x_b - x_c|\,\}, & \text{if } \angle C \text{ opens horizontally} \\ a + b - 2\min\{\,|y_a - y_c|\,, |y_b - y_c|\,\}, & \text{if } \angle C \text{ opens vertically} \end{cases} \tag{6.7.1}$$

PROOF:

Case 1: $\angle C$ is properly oriented. Then by the taxicab distance formula $c = a + b$. [Note that in this case, the x-coordinate of point C will be the same as the x-coordinate of either point A or point B, and similarly for its y-coordinate. Therefore, the minimum of $|x_a - x_c|$ and $|x_b - x_c|$ will be zero, and the minimum of $|y_a - y_c|$ and $|y_b - y_c|$ will be zero, which causes both the horizontal and vertical expressions in Equation 6.7.1 to simplify to $c = a + b$. So, the properly oriented case is really a special case of both the horizontal and vertical cases.]

Case 2: $\angle C$ opens horizontally (Figure 6.7.1). Then, the length of the hypotenuse is

$$\begin{aligned} c &= d(A, B) \\ &= (b_2 - a_2) + (b_1 + a_1) \\ &= b_2 - a_2 + b_1 + a_1 + a_2 - a_2 \\ &= (a_1 + a_2) + (b_1 + b_2) - 2a_2 \\ &= a + b - 2\,|x_b - x_c| \end{aligned}$$

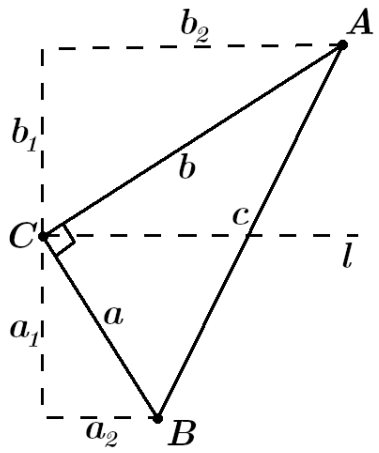


Figure 6.7.1 Deriving the Pythagorean Theorem in taxicab geometry for a right triangle with right angle opening horizontally

If the labels for points A and B are interchanged, then $c = a + b - 2|x_a - x_c|$. In any case, the smaller of $|x_a - x_c|$ and $|x_b - x_c|$ is being chosen, and so the result follows with the **min** function properly selecting between these two cases.

Case 3: $\angle C$ opens vertically (Figure 6.7.2). Then, the length of the hypotenuse is

$$\begin{aligned} c &= d(A, B) \\ &= (b_1 - a_1) + (a_2 + b_2) \\ &= b_1 - a_1 + a_2 + b_2 + a_1 - a_1 \\ &= (a_1 + a_2) + (b_1 + b_2) - 2a_1 \\ &= a + b - 2|y_b - y_c| \end{aligned}$$

As in Case 2, if the labels for points A and B are interchanged, then $c = a + b - 2|y_a - y_c|$. The smaller of $|x_a - x_c|$ and $|x_b - x_c|$ is again being chosen, and so the result follows with the **min** function properly selecting between these two cases. □

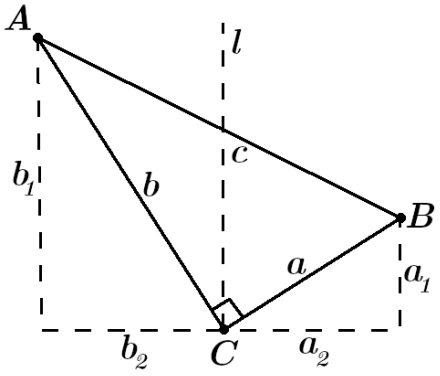


Figure 6.7.2 Deriving the Pythagorean Theorem in taxicab geometry for a right triangle with right angle opening vertically

Section 6.8: Law of Cosines

The Law of Cosines is a generalization of the Pythagorean Theorem to non-right triangles. In this section we explore what the Law of Cosines looks like in taxicab geometry.

[Note that the Law of Cosines research in the paper "On Trigonometric Functions and Cosine and Sine Rules in Taxicab Plane" by Bayar, Ekmekçi, and Özcan uses Euclidean angles and so is not a development of this Law for *pure taxicab geometry*.]

Lemma 6.8.1 *If* $\triangle ABC$ *has a side* $\overline{AB}$ *(of length* c*) parallel to a coordinate axis with both* $\angle A$ *and* $\angle B$ *acute, then*

$$a = b + c - 2b\cos A$$
$$b = a + c - 2a\cos B$$
$$c = \frac{a^2 + b^2 - 2ab\cos C}{a + b}$$

PROOF: Without loss of generality, assume side $\overline{AB}$ is parallel to the x-axis. Drop an altitude of length h from C through the triangle to point H on side $\overline{AB}$ and draw a taxicab circle (gray) of radius b about the point A (Figure 6.8.1). Then by the Arc Length Theorem (Theorem 4.1.4), the arc length subtended on the circle by $\angle A$ is $s = b \cdot m(\angle A)$. Since the circle is of radius b, the length of s is also $s = b - p + h$ where $p = d(A, H)$. Setting these equal gives

$$b - p + h = b \cdot m(\angle A)$$
$$b\left(1 - m(\angle A)\right) = p - h$$

Using the right triangle CHB, $a = h + (c - p)$ or equivalently $p - h = c - a$. Substituting this into the expression above gives

$$b\left(1 - m(\angle A)\right) = c - a$$
$$m(\angle A) = \frac{a + b - c}{b}$$

By Equation 4.3.1,

$$\cos A = 1 - \frac{1}{2}m(\angle A)$$
$$= 1 - \frac{1}{2}\frac{a + b - c}{b}$$

Solving this for a gives $a = b + c - 2b\cos A$.

A similar process focused on point B yields $b = a + c - 2a\cos B$.

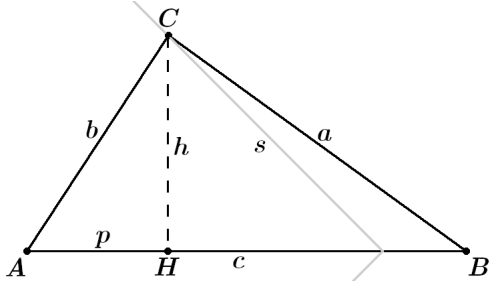


Figure 6.8.1 A special case of the Law of Cosines with a side parallel to a coordinate axis and associated angles acute.

Since the sum of the angles of a taxicab triangle is 4 t-radians, $C = 4 - A - B$ and thus

$$\begin{aligned}
\cos C &= \cos\big(4 + - (A + B)\big) \\
&= -\cos(A + B) && \text{(Table 4.3.1)} \\
&= 1 - \big|\cos(A) + \cos(B)\big| && \text{(Theorem 4.4.1)} \\
&= 1 - \cos(A) - \cos(B) && (\cos(A) > 0, \cos(B) > 0) \\
&= 1 + \frac{a - b - c}{2b} + \frac{b - a - c}{2a} \\
&= \frac{a^2 + b^2 - ac - bc}{2ab}
\end{aligned}$$

where the cosine functions were replaced with expressions found earlier in this proof. Solving this for c gives

$$c = \frac{a^2 + b^2 - 2ab\cos C}{a + b}$$ □

Note that if, for instance, $\angle A$ is a right angle in Lemma 6.8.1, then the first formula reduces to the taxicab Pythagorean Theorem for a properly oriented right angle (with the hypotenuse labeled a), the second to $\cos(B) = \frac{c}{a}$ which can be shown to be equal to $1 - \frac{1}{2}m(\angle B)$, and the third to $\cos(C) = 1 - \frac{c}{a}$ which can be shown to be equal to $1 - \frac{1}{2}m(\angle C)$, which is what we would expect for a properly oriented right triangle.

Lemma 6.8.2 *If* $\triangle ABC$ *has a side* $\overline{AB}$ *parallel to a coordinate axis with* $\angle A$ *obtuse, then* $a = b + c$.

PROOF: From Figure 6.8.2 it is clear from $\triangle CHA$ that $b = p + h$ and from $\triangle CHB$ that $(c + p) + h = a$. Substituting the first equation into the seconds yields $a = b + c$. □

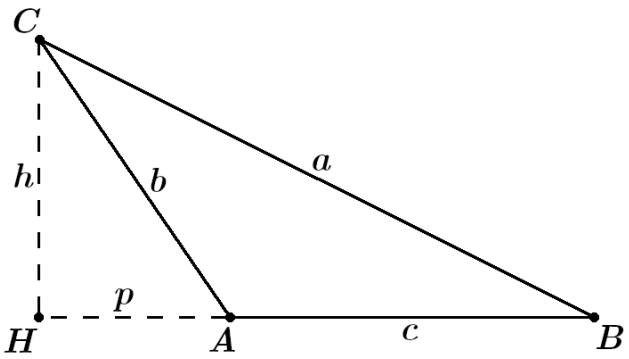


Figure 6.8.2 A special case of the Law of Cosines with a side parallel to a coordinate axis and an associated angle obtuse.

Section 6.9: Law of Sines

Since an ASASA relationship between two triangles is insufficient to guarantee they are congruent in taxicab geometry (Section 6.2), it is not surprising that the Law of Sines fails to carry over from Euclidean geometry to taxicab geometry. But, we can recover the Law of Sines in some situations and find some alternate forms of it in others.

[Note that the Law of Sines research in the paper "On Trigonometric Functions and Cosine and Sine Rules in Taxicab Plane" by Bayar, Ekmekçi, and Özcan uses Euclidean angles and so is not a development of this Law for *pure taxicab geometry*.]

Lemma 6.9.1 *If* $\triangle ABC$ *has a side* $\overline{AB}$ *parallel to a coordinate axis with both* $\angle A$ *and* $\angle B$ *acute, then*

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

PROOF: As noted in Section 4.3, for a right triangle with a properly oriented right angle the usual construction of sine as "opposite over hypotenuse" from Euclidean geometry still holds in taxicab geometry. Therefore, using Figure 6.9.1, we have

$$\sin(A) = \frac{h}{b} \text{ and } \sin(B) = \frac{h}{a}$$

Solving these for h, setting equal, and rearranging yields

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

□

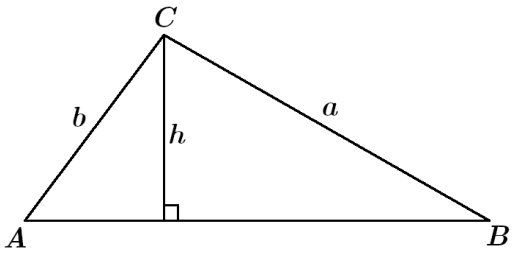


Figure 6.9.1 A triangle with one side parallel to a coordinate axis and acute associated angles

Section 6.10: Taxicab Excircles

Excircles are at the heart of a version of the Apollonius circle theorem that will be investigated in Section 6.11. In this section we determine the conditions under which excircles exist for all sides of taxicab triangle which will guide the search for Apollonius circles.

Material in this section will rely heavily on inscribed angles (Section 4.2) and inscribed triangles (Section 6.3). The reader is encouraged to be comfortable with these concepts and results before proceeding.

Definition 6.10.1 *An **excircle** ("escribed circle") of a triangle is a circle outside the triangle that is tangent to a side of the triangle and tangent to the rays that form the angle opposite the side in the triangle.*

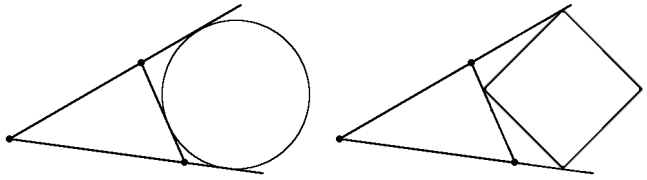


Figure 6.10.1 Excircles in Euclidean and taxicab geometry

In Euclidean geometry, an excircle exists for every side of any triangle. This is not the case in taxicab geometry.

Lemma 6.10.1 *The side of a triangle opposite a taxicab angle that is not inscribed does not have an excircle.*

PROOF: Without loss of generality, anchor one ray of the angle at the origin and place it in the upper half of Quadrant I (the black ray in Figure 6.10.2). As explained in Section 4.2, an angle that is not inscribed will always fully encompass a right angle aligned along lines of slope ± 1, so the black ray in Quadrant IV is of the kind necessary in this situation. When two points on the rays are connected to form the side of the triangle opposite the angle, any taxicab circle (green) tangent at its left point to the side of the triangle will never intersect the black rays because an angle that is not inscribed opens too widely. This is true no matter how the rays are connected to form the side of the triangle. The sides of the taxicab circle will simply not be steep enough to intersect either of the diverging rays to form an excircle. In addition, any attempt to draw taxicab circles at the vertices will manage to touch one of the black rays but not the other (imagine expanding the red taxicab circles). Therefore, an excircle cannot be formed at a side of a triangle that is opposite an angle that is not inscribed. □

Note that the failure of an angle that is not inscribed to have an excircle provides a grounded concept for Theorem 2.1 in the Ermis research as the slope conditions of that theorem simply describe a triangle containing an angle that is not inscribed.

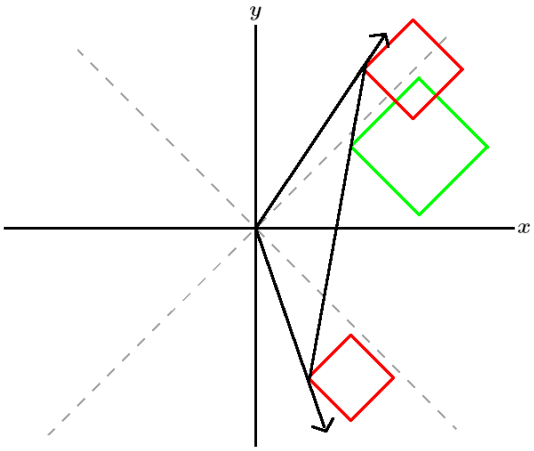


Figure 6.10.2 An angle that is not inscribed (black rays) opens too widely to allow the construction of an excircle on the side opposite the angle (green taxicab circle) or at the vertices (red taxicab circles).

Lemma 6.10.1 indicates that the search for excircles must be constrained to triangles whose angles are all inscribed. This is the precise definition an inscribed triangle (Definition 6.3.1). We will examine the classes of inscribed triangles separately based on the number of completely inscribed angles in the triangle. We will see that it is only possible to construct excircles on all sides of a triangle for certain kinds of inscribed triangles.

Lemma 6.10.2 *Excircles can be constructed at all sides of a minimally inscribed triangle.*

PROOF: Let $\triangle ABC$ be a minimally inscribed triangle (Definition 6.3.2) with completely inscribed $\angle A$, strictly negatively inscribed $\angle B$, and strictly positively inscribed $\angle C$ as shown in Figure 6.10.3. Since $\angle A$ is completely inscribed, in this case the absolute slope of its rays is less than 1; therefore, a taxicab circle in its interior (blue) can be expanded to be tangent to both of the angle's rays. Using the gray dotted lines at $\angle B$ and $\angle C$ we can see that the strictly inscribed nature of the angles limits the slope possibilities for $\overline{BC}$ in such a way that taxicab circles outside the triangle and tangent to $\overline{BC}$ will remain outside the triangle. Therefore, $\overline{BC}$ has an excircle.

Taxicab circles tangent to the outside of $\overline{AB}$ can never be tangent to extensions of $\overline{AC}$ unless the bottom vertex of the taxicab circle is actually at point A (green), but such circles will always be able to be tangent to extensions of $\overline{BC}$ if of sufficient size. Similarly, the excircle for $\overline{AC}$ must have its top vertex at point A (red).

Triangles with other orientations of the completely inscribed angle are proven similarly. □

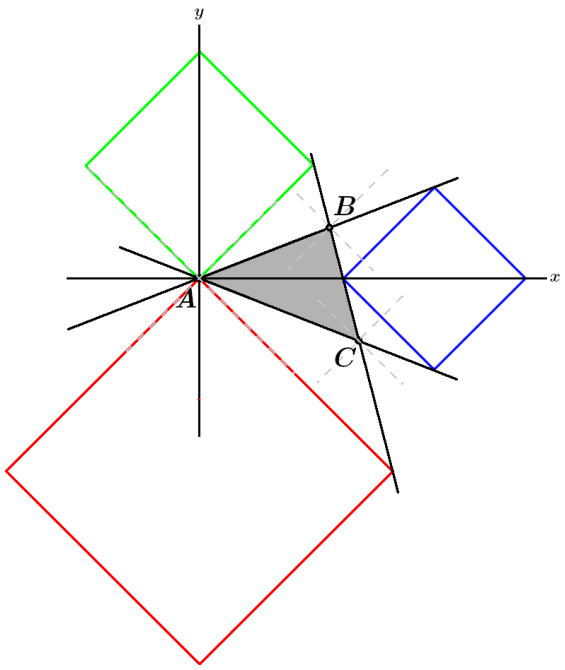


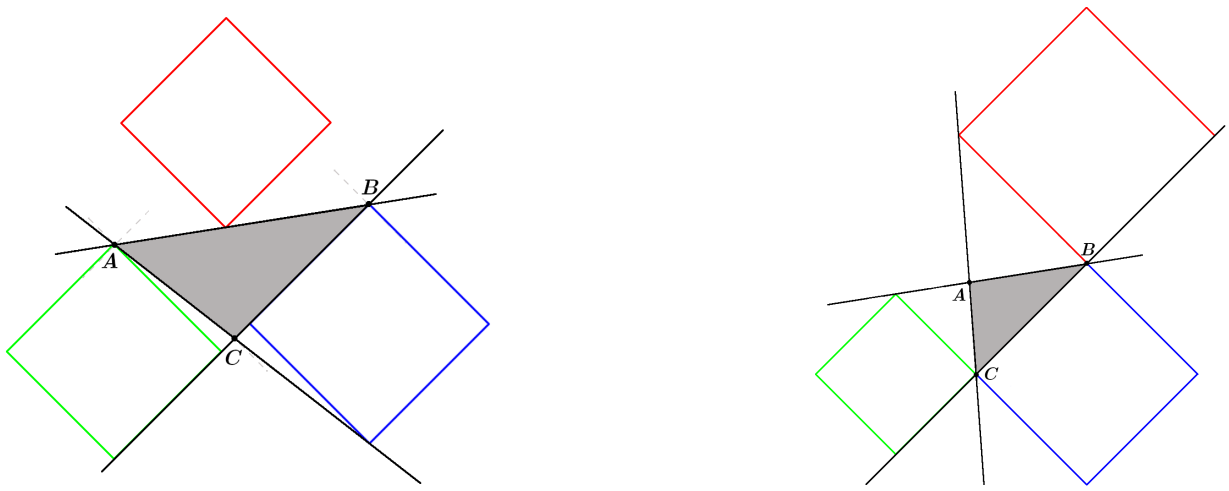
Figure 6.10.3 Constructing taxicab excircles (green, blue, and red) for a minimally inscribed triangle (exactly one completely inscribed angle).

Lemma 6.10.3 *Excircles can be constructed at all sides of an inscribed taxicab triangle that has two completely inscribed angles only if for the two non-diagonal sides one side is steep and the other side is shallow.*

PROOF: By Theorem 6.3.4, triangles with two completely inscribed angles have one side with slope ± 1; choose $\overline{BC}$ (i.e. $|m_a| = 1$).

Case 1: Both sides shallow ($|m_b| < 1$ and $|m_c| < 1$) as in Figure 6.10.4a. Any possible excircle placed along $\overline{AB}$ (red) will never intersect extensions of both $\overline{AC}$ and $\overline{BC}$ simultaneously. Since the slope of the proposed excircle is greater than or equal to the slopes of both lines, only by placing the circle at a vertex can one side be touched, but then the other side will not be touched. Therefore, $\overline{AB}$ cannot have an excircle tangent to it. The case of both sides steep is proved similarly.

Case 2: $|m_b| > 1$ and $|m_c| < 1$ (Figure 6.10.4b). With $\overline{AC}$ now steep, a circle can be placed with its bottom vertex at B and inflated to eventually intersect the extension of $\overline{AC}$ thus forming an excircle for $\overline{AB}$. The excircle for $\overline{BC}$ exists due to its diagonal nature and the excircle for $\overline{AC}$ exists due to the shallow slope of $\overline{AB}$. □



a) b)

Figure 6.10.4 Attempting to construct taxicab excircles (green, blue, and red) for a triangle having two completely inscribed angles in a) a failing case and b) a successful case.

Lemma 6.10.4 *Excircles cannot be constructed at all sides of an inscribed taxicab triangle that has three completely inscribed angles.*

PROOF: Two of the sides opposite the completely inscribed angles must have slope ± 1 (Theorem 6.3.3). This type of triangle will suffer the same problem as the failing case of Lemma 6.10.3. Figure 6.10.4a just needs to be subtly adjusted so that $\angle C$ is a right angle. For the side opposite the right angle, the extensions of the triangle's other sides will diverge away parallel to any possible excircle that could be drawn, and a circle at either vertex will only intersect one of the side extensions. Therefore, the triangle will only have two excircles. □

Section 6.11: Taxicab Apollonius Circle for a Triangle

[Introductory Note: Apollonius circles for taxicab triangles were first investigated in 2018 by Ermis, Gelisgen, and Ekici. But, the authors did not incorporate the concepts of inscribed angles and triangles. Frequent mention of their research will be made in this section to validate that the approach using inscribed triangles arrives at the same conclusions with the exception of an error in the proof of one of their theorems which is herein corrected.]

A general problem addressed by the ancient geometer Apollonius was the construction of a circle which is tangent to three other objects, typically points, lines, or circles. If the other objects are taken as circles, a specific version of this problem can be stated as: "Is there the circle that tangentially encompasses all three excircles of a given triangle?" The answer in Euclidean geometry is, "Yes." The answer in taxicab geometry is ... "Rarely."

While the search for taxicab excircles had a relatively clean result in terms of inscribed angles, the existence of an Apollonius circle for those excircles is fraught with an additional complication. We will take a simple example of a triangle with a vertical side and a horizontal side to illustrate this complication.

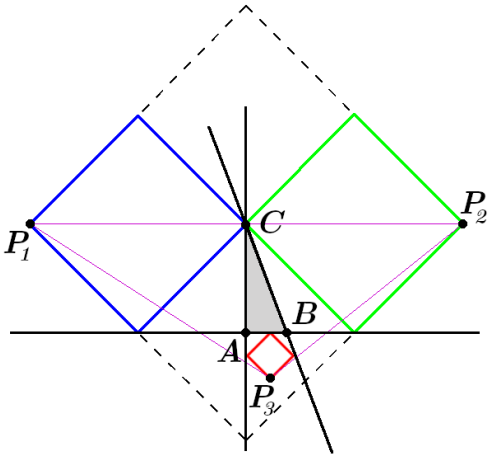


Figure 6.11.1 A triangle whose taxicab excircles do not have an enclosing tangential taxicab (Apollonius) circle

The triangle shown in Figure 6.11.1 fits the conditions of Lemma 6.10.2 and so it has a full complement of excircles. But, the side $\overline{BC}$ is very steep. This causes the excircle for side $\overline{AB}$ (red) to be squeezed between the extensions of $\overline{AC}$ and $\overline{BC}$ to the point that it is in a sense "interior" to the other two excircles (blue and green). The effect of this is that any circle tangent to the excircles for $\overline{AC}$ and $\overline{BC}$ (dotted black line) will not be tangent to the excircle for $\overline{AB}$; the third small excircle will always be on the interior of the enclosing circle.

The technical reason that a taxicab Apollonius circle does not exist in this case is as follows. Choose points P_1, P_2, and P_3, one on each excircle as shown. Then $\angle P_1P_3P_2$ is not inscribed (it fully encompasses a right angle aligned along lines of slope ± 1) and therefore $\triangle P_1P_3P_2$ (purple) is not inscribed. By the Three-Point Circle Theorem (Corollary 6.5.2) there is no taxicab circle passing through these three points.

If the absolute slope of $\overline{BC}$ is reduced sufficiently, the enclosing circle can be made tangent (Figure 6.11.2). But, if the absolute slope is reduced to unity, then the triangle will have three inscribed angles and therefore fail to have a full complement of excircles. So, there is a bounded range for the slope of $\overline{BC}$ in which an Apollonius circle exists.

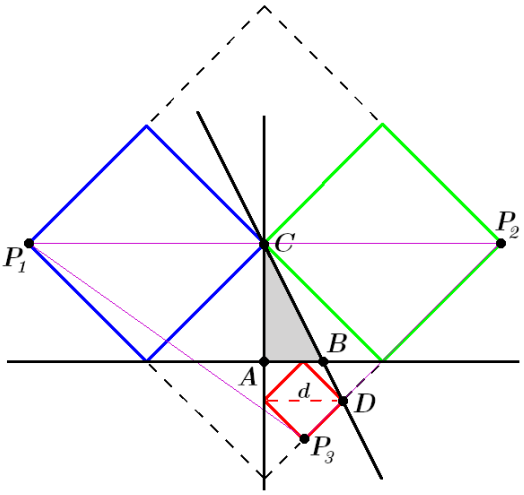


Figure 6.11.2 A triangle whose taxicab excircles (barely) have an enclosing tangential taxicab Apollonius circle

To find the slope range for $\overline{BC}$, notice that the point D in Figure 6.11.2 is the crucial point. In order for $\angle P_1P_3P_2$ to be inscribed (so that the points form an inscribed triangle and the Three-Point Circle Theorem can be applied to construct the Apollonius circle), $D = \left(d, -\frac{1}{2}d\right)$ must lie on the extension of $\overline{BC}$ and *on or to the right* of the extension of the lower right section of the excircle of side $\overline{BC}$ (green). Using the notation (x_*, y_*) for the coordinates of the vertices of the triangle, the equation for the extension of the lower right line of the excircle for side $\overline{BC}$ is $y = x - y_c$ which for point D leads to the constraint

$$-\frac{1}{2}d \le d - y_c$$

$$y_c \le \frac{3}{2}d$$

The equation for the line through $\overline{BC}$ with slope m_a is $y = -m_a x + y_c$ which when combined with the inequality above leads to the constraint

$$-\frac{1}{2}d = m_a d + y_c$$

$$y_c = -(m_a + \frac{1}{2})d \le \frac{3}{2}d$$

$$m_a \ge -2$$

Therefore, in this orientation, the slope of $\overline{BC}$ is constrained to $-2 \le m_a < -1$ in order for the Apollonius circle to exist. This analysis provides a template for proving all four cases (hypotenuse slope steep vs. shallow and positive vs. negative) of the following lemma and also outlines the kind of strategy necessary for handling other types of triangles leading up to a general Triangle Apollonius Circle Theorem.

Lemma 6.11.1 *A triangle with a horizontal side and a vertical side has a taxicab Apollonius circle tangentially encompassing its taxicab excircles if the slope* m *of the side opposite the right angle satisfies* $\frac{1}{2} \leq |m| < 1$ *or* $1 < |m| \leq 2$.

[Note that this result agrees with Corollary 2.1 in the Ermis research.]

Before moving on to generally deal with triangles having other orientations and shapes, it should be demonstrated that the conclusions in the Ermis Apollonius research are not quite right. Consider the triangle formed by lines of slope $\frac{3}{4}$, $\frac{1}{5}$, and 3 shown in Figure 6.11.3 with its Apollonius circle (orange). The value of ρ in Ermis Theorem 2.2 for this triangle is -0.1. But the triangle satisfies branch 2 of part i) of this theorem which indicates that ρ must be positive for the taxicab Apollonius circle to exist. The Ermis theorem is further afflicted in that interchanging the points A and B leads to $\rho = -10$.

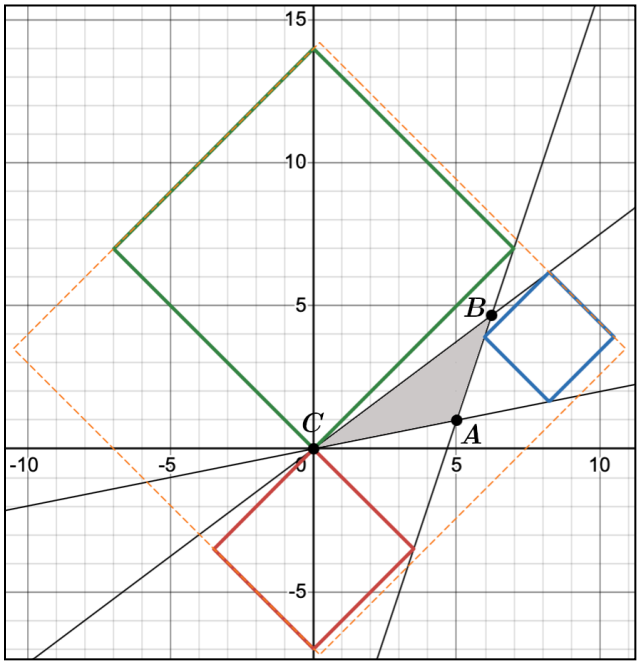


Figure 6.11.3 Example of a triangle with a taxicab Apollonius circle (orange) that violates the conditions of Theorem 2.2 of the Ermis paper

We will now attempt to find an Apollonius circle for a more general triangle. Consider a triangle formed by two shallow lines with same signed slope as in Figure 6.11.4, assuming without loss of generality that $0 \leq m_b < m_a < 1$. The $\angle C$ formed at the intersection of the lines is completely inscribed.

Before continuing, consider any triangle with one completely inscribed angle formed by two lines whose slopes are both steep or both shallow and have the same sign. Such a triangle is always equivalent to a triangle positioned in the first quadrant. If the axes are relabeled, or equivalently if the triangle is reflected about the lines $y = \pm x$ and/or $x = 0$ and/or $y = 0$, the triangle can be transformed to the lower half of the first quadrant with the completely inscribed angle at the origin. Therefore, our analysis will be limited to the possible cases in the lower half of the first quadrant, like Figure 6.11.4.

In order for excircles to exist, there are constraints on the slope m_c of side $\overline{AB}$. If $0 < m_c < 1$ (purple dashed line) then $\angle B$ will be a second completely inscribed angle but the triangle will not satisfy Lemma 6.10.3 and therefore will not have a full complement of excircles. If $-1 < m_c < 0$ (other purple dashed line) then $\angle A$ is a second completely inscribed angle, but again the triangle will not satisfy Lemma 6.10.3 and will not have a full complement of excircles. So, we must in the end include the restriction $|m_c| > 1$. Note that the triangle is minimally inscribed (Definition 6.3.2).

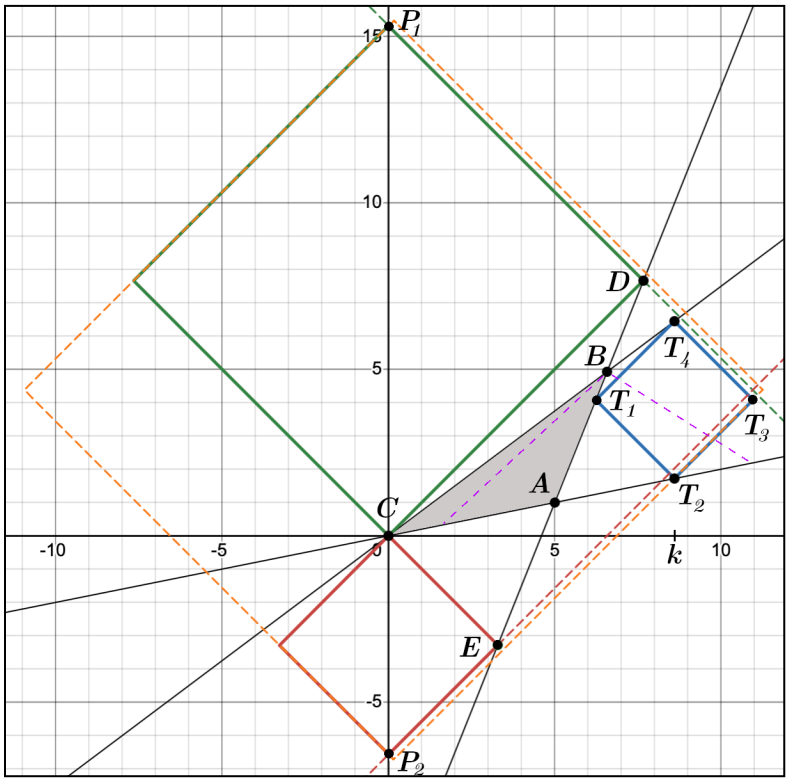


Figure 6.11.4 Constructing taxicab excircles for a triangle contained in one quadrant with one completely inscribed angle and a positive steep-sloped opposite side.

Using Figure 6.11.4 as a guide, the radius of the excircle for $\overline{AB}$ is $r = \frac{1}{2}k(m_a - m_b)$ and so the coordinates of the vertices of the excircle are:

$$T_2 = (k, m_b k)$$
$$T_4 = (k, m_a k)$$
$$T_1 = (k - r, m_a k - r) = \left(k\left(1 - \frac{1}{2}m_a + \frac{1}{2}m_b\right), \frac{k}{2}\left(m_a + m_b\right)\right)$$
$$T_3 = (k + r, m_a k - r) = \left(k\left(1 + \frac{1}{2}m_a - \frac{1}{2}m_b\right), \frac{k}{2}\left(m_a + m_b\right)\right)$$

Since T_1 is on the side $\overline{AB}$ with equation $y = m_c(x - x_a) + y_a$, we can substitute its coordinates and solve for k to get

$$k = \frac{2(y_a - x_a m_c)}{m_a + m_b + m_a m_c - m_b m_c - 2m_c}$$

Now, $\angle P_1T_3P_2$ must be inscribed in order for the three points to lie on a taxicab circle (Three-Point Circle Theorem - Corollary 6.5.2). In order for this to be the case, T_3 cannot be interior to the corner formed by the red and green extensions of the excircles for $\overline{AC}$ and $\overline{BC}$. And this is where the mistake occurs in the proof of Ermis Theorem 2.2. The condition applied in the Ermis paper forces T_3 to be BOTH above the green dotted line AND below the red dotted line when only one of these conditions is necessary for $\angle P_1T_3P_2$ to be inscribed.

Taking individually the constraints that yield an Apollonius circle, first consider the dashed green line. Point D is the intersection of $y = x$ with the extension of $\overline{AB}$ with equation $y = m_c(x - x_a) + y_a$. This yields

$$D = \left(\frac{y_a - x_a m_c}{1 - m_c}, \frac{y_a - x_a m_c}{1 - m_c} \right)$$

So, the equation of the green dashed line is

$$y = -x + \frac{2(y_a - x_a m_c)}{1 - m_c}$$

For T_3 to be above this line, T_4 would have to be as well. Using this point for its simpler coordinates, its y-value must be greater than the equation for the green line:

$$m_a k \geq -k + \frac{2(y_a - x_a m_c)}{1 - m_c}$$

Noting that $m_a + 1 > 0$, solving for k gives

$$k \geq \frac{2(y_a - x_a m_c)}{(1 - m_c)(m_a + 1)}$$

Substituting for k:

$$\frac{2(y_a - x_a m_c)}{m_a + m_b + m_a m_c - m_b m_c - 2m_c} \geq \frac{2(y_a - x_a m_c)}{(1 - m_c)(m_a + 1)}$$

Since $k > 0$ in this situation, the left denominator must have the same sign as the left numerator. For $m_c > 1$ all fraction parts are negative; for $m_c < -1$ all fraction parts are positive. Taking this into account, in either case after simplification we have:

$$\frac{m_b}{m_c} + 2m_a - m_b - 1 \geq \frac{1}{m_c}$$

This is one condition that a triangle with $0 \leq m_b < m_a < 1$ can satisfy in order to have a taxicab Apollonius circle.

For the alternate constraint, point T_2 would have to be below the dashed red line. The point E is the intersection of $y = -x$ with $y = m_c(x - x_a) + y_a$ which yields

$$E = \left(\frac{x_a m_c - y_a}{m_c + 1}, \frac{y_a - x_a m_c}{m_c + 1} \right)$$

The equation of the line is therefore

$$y = x + \frac{2(y_a - x_a m_c)}{m_c + 1}$$

So, with T_2 below the line we have

$$m_b k \le k + \frac{2(y_a - x_a m_c)}{m_c + 1}$$

Solving for k ($m_b - 1 < 0$) and substituting gives

$$\frac{2(y_a - x_a m_c)}{m_a + m_b + m_a m_c - m_b m_c - 2m_c} \ge \frac{2(y_a - x_a m_c)}{(m_c + 1)(m_b - 1)}$$

The sign analysis has not changed so simplifying yields

$$1 - m_a + 2m_b - \frac{m_a}{m_c} \le \frac{1}{m_c}$$

This is the second condition that a triangle with $0 \le m_b < m_a < 1$ can satisfy in order to have a taxicab Apollonius circle. Note that combining the two constraints yields the overly restrictive function ρ in Ermis Theorem 2.2.

As an example, the triangle in Figure 6.11.4 has slopes $m_a = \frac{3}{4}$, $m_b = \frac{1}{5}$, and $m_c = \frac{5}{2}$. It satisfies the second constraint ($0.35 \le 0.4$) but not the first ($0.38 \not\ge 0.4$).

For a minimally inscribed triangle, the only remaining case is when the signs of m_a and m_b differ, namely $-1 < m_b < 0 < m_a < 1$. But, even in this case the items of consequence ($m_a + 1 > 0$, $m_b - 1 < 0$, and $k > 0$) in the analysis still hold up even if a relabeling of the points must occur. Therefore, we have the following results.

Lemma 6.11.2 *A minimally inscribed triangle with shallow slope sides $0 < m_a < 1$ and $-1 < m_b < m_a$ forming the completely inscribed $\angle C$ has an Apollonius circle tangentially encompassing its taxicab excircles if one of the following conditions holds:*

for $1 < |m_c| < \infty$: $\quad \dfrac{m_b}{m_c} + 2m_a - m_b - 1 \ge \dfrac{1}{m_c}$ $\quad$ *or* $\quad 1 - m_a + 2m_b - \dfrac{m_a}{m_c} \le \dfrac{1}{m_c}$

for $\overline{AB}$ vertical: $\quad 2m_a - m_b \ge 1$ $\quad$ *or* $\quad m_a - 2m_b \ge 1$

where m_a and m_b are the slopes of the sides opposite $\angle A$ and $\angle B$, respectively.

To establish the constraint for the $\overline{AB}$ vertical case let $m_c \to \pm\infty$ in the non-vertical case. Then the first constraint simplifies to $2m_a - m_b \ge 1$ and the second to $m_a - 2m_b \ge 1$. Also note that by setting $m_b = 0$ we see that Lemma 6.11.1 is a special case of Lemma 6.11.2 when limited to the first quadrant.

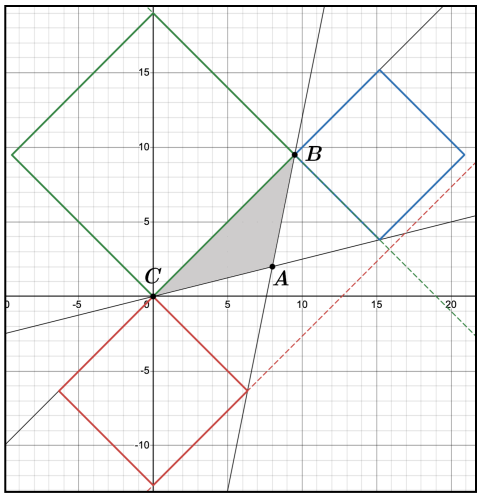


Figure 6.11.5 Taxicab excircles for a triangle with two completely inscribed angles

For a triangle with two completely inscribed angles, we already know that the triangle must have one side with slope ± 1 (Theorem 6.3.4) and by Lemma 6.10.3 the other sides must have opposing slopes - one shallow and one steep. Assume $|m_b| < 1$ and $|m_c| > 1$. The basic constraint process outlined for minimally inscribed triangles works here as well. If we simply apply the first constraint of Lemma 6.11.2 with $m_a = 1$ we find that $m_c \geq 1$ (for $m_c > 1$) or $m_c \leq 1$ (for $m_c < -1$) which we have already required in order to even have a full complement of excircles. Visually, in Figure 6.11.5, the side $\overline{AB}$ can have any slope $|m_c| > 1$ and the excircles for $\overline{AB}$ (blue) and $\overline{BC}$ (green) will be unaffected leaving the excircle for $\overline{AB}$ always to the right of the extension of the excircle for $\overline{BC}$, which is exactly what the first constraint of Lemma 6.11.2 is testing. Therefore, a triangle with two completely inscribed angles that has a full complement of excircles will always have a taxicab Apollonius circle.

Theorem 6.11.3 (Triangle Apollonius Circle Theorem) *A* $\triangle ABC$ *has a taxicab Apollonius circle if it is transformationally equivalent to a triangle outlined in the lemmas in this section, namely:*

1) *a minimally inscribed triangle with shallow slope sides* $0 < m_a < 1$ *and* $-1 < m_b < m_a$ *forming the completely inscribed* $\angle C$ *with either*

   a. $1 < |m_c| < \infty$ *and either* $\dfrac{m_b}{m_c} + 2m_a - m_b - 1 \geq \dfrac{1}{m_c}$ *or* $1 - m_a + 2m_b - \dfrac{m_a}{m_c} \leq \dfrac{1}{m_c}$, *or*

   b. $\overline{AB}$ *is vertical and* $2m_a - m_b \geq 1$

2) *a triangle with exactly two completely inscribed angles and non-diagonal sides with opposing slopes (one shallow, one steep)*

Apollonius Points

In Euclidean geometry, the Apollonius circle is tangent to each excircle at a point. If these points are connected to the vertex associated with their excircle, the three lines intersect at a point called the Apollonius point. Since the taxicab Apollonius circle is tangent to most excircles along a line segment the concept of an Apollonius point does not exist in taxicab geometry.

Before concluding this section, let's examine a general case where a taxicab Apollonius circle fails to exist. The triangle in Figure 6.11.6 has $m_a = \frac{3}{5}$, $m_b = \frac{3}{10}$, and $m_c = -8$. It satisfies neither the first constraint ($-0.1375 \not\geq -0.125$) where the excircle for $\overline{AB}$ is narrowly below the dashed green line nor the second ($1.075 \not\leq -0.125$) where is it far above the dashed red line. The points P_1, P_2, and P_3 fail to form an inscribed triangle as was described in the lead up to the development of the constraints and so a taxicab circle through the points does not exist.

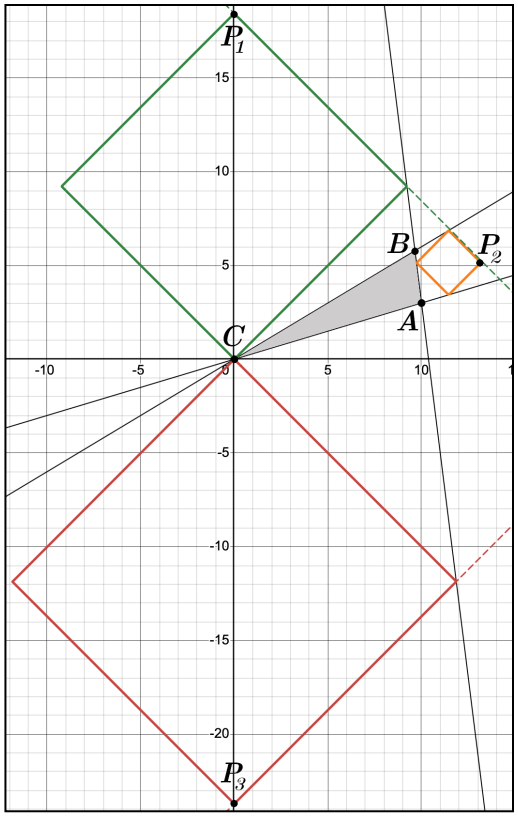


Figure 6.11.6 A triangle with a full complement of taxicab excircles that fails to have a taxicab Apollonius circle

Other interesting questions can be asked such as: for which values of m_a and m_b will any choice of m_c lead to an Apollonius circle, or a lack of an Apollonius circle? Solving the first and second constraints for $m_c > 1$ in Lemma 6.11.2 we get

$$m_c \geq \frac{1 - m_b}{2m_a - m_b - 1} \quad \text{and} \quad m_c \leq \frac{m_a + 1}{1 - m_a + 2m_b}$$

where we must take care to consider only $2m_a - m_b - 1 > 0$ for the moment. If we wish to find where there is an Apollonius circle for any choice of $m_c > 1$ then we would want

$$\frac{1 - m_b}{2m_a - m_b - 1} < \frac{m_a + 1}{1 - m_a + 2m_b}$$

When this inequality is graphed with $2m_a - m_b - 1 > 0$ the gray shaded region in Figure 6.11.7 results. The point marked is for the slopes from Figure 6.11.4 which is just inside the shaded region. This illustrates that we could have chosen any value $m_c > 1$ and the triangle would have had a taxicab Apollonius circle.

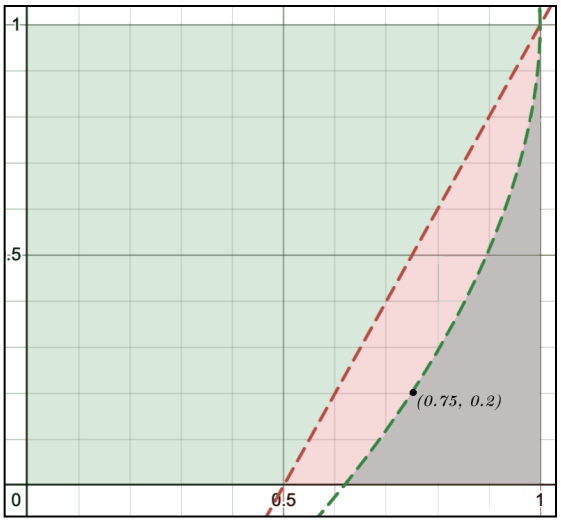


Figure 6.11.7 A region (gray) of values for m_a (x-axis) and m_b (y-axis) in which any choice of m_c would lead to a taxicab Apollonius circle for the triangle

As mentioned at the beginning of this section, Apollonius circles for taxicab triangles were first investigated in 2018 by Ermis, Gelisgen, and Ekici. Their approach was very direct and the existence of an Apollonius circle for a triangle was conditioned on the slopes of the lines forming the triangle. This is similar to what has been presented here. However, that original work did not clearly separate out the conditions for the existence of a full complement of excircles (as presented in Section 6.10), did not make it abundantly clear how and why excircles could exist but an Apollonius circle not exist, required a slightly too strict condition for the existence of some Apollonius circles, and did not incorporate the concepts of inscribed angles and triangles. It is hoped that the expansion, elaboration, and slight correction of their work presented in these two sections has made the concept and existence of taxicab excircles and Apollonius circles clearer.

Excenters of a Triangle

If the centers of the excircles of a triangle (the excenters) are connected to their corresponding vertices, the three lines intersect at a common point inside the triangle (Figure 6.11.8).

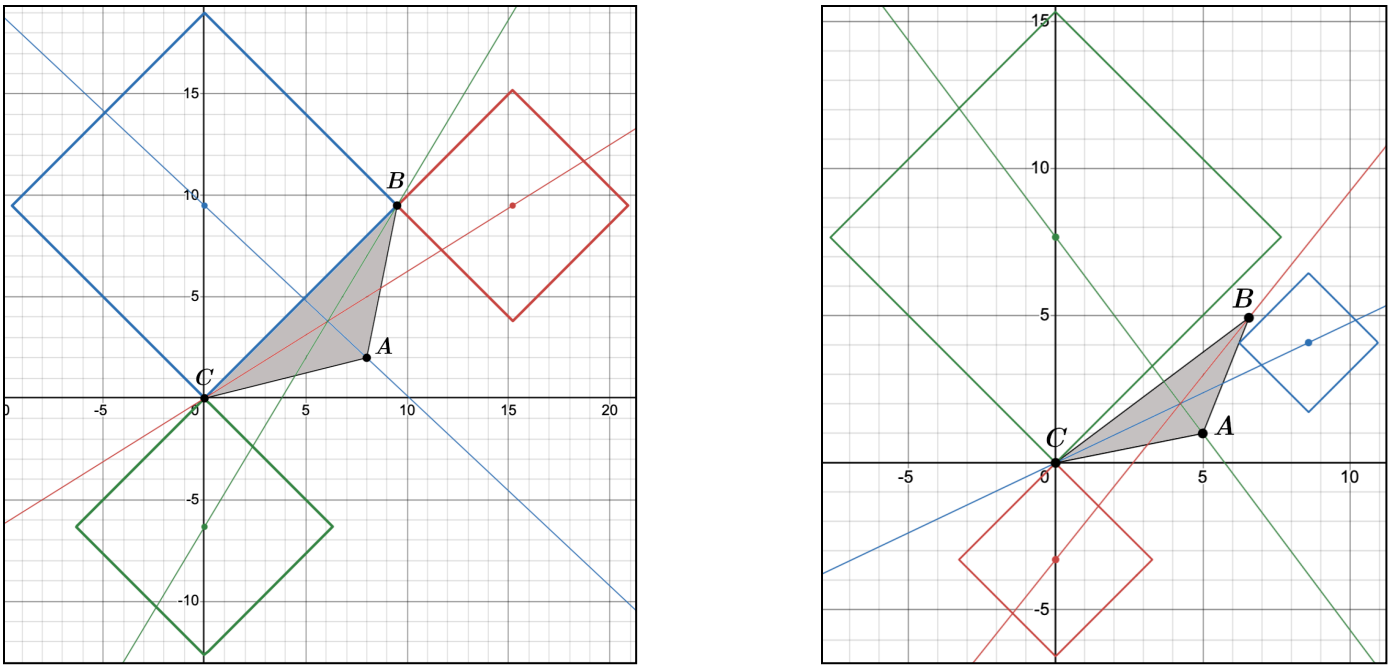


Figure 6.11.8 Lines connecting the excenters of a triangle to their corresponding vertices are concurrent

Section 6.12: Nagel Points

The Nagel point of a triangle occurs at the intersection of the three lines connecting the vertices to their associated extouch point on the side opposite the vertex. An extouch point is where the excircle touches the side of the triangle. Nagel points do not always exist in taxicab geometry and are much less interesting than in Euclidean geometry.

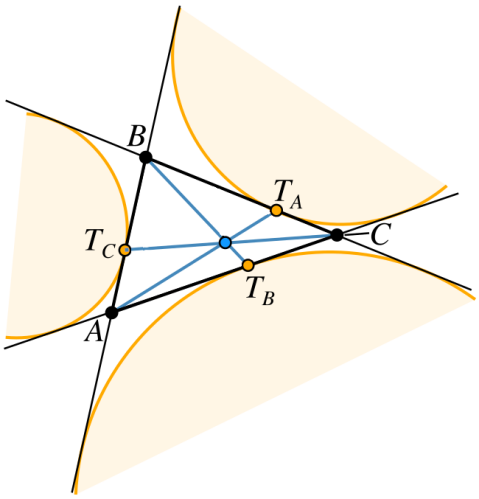


Figure 6.12.1 The Nagel point (blue) for a Euclidean triangle at the intersection of the lines connecting the vertices to the extouch points T_* where the excircles touch the sides of the triangle

Minimally Inscribed Triangles

A minimally inscribed triangle always has a full complement of excircles (Lemma 6.10.2). Two of these excircles always touch the triangle at the vertex of the completely inscribed angle as in Figure 6.10.3. So, the lines connecting vertices B and C to their extouch points are merely the sides $\overline{AB}$ and $\overline{AC}$ of the triangle intersecting at point A. Connecting vertex A to its extouch point simply creates another line through point A. Therefore, the Nagel point of a minimally inscribed triangle is the vertex of the completely inscribed angle.

Triangles with Two Completely Inscribed Angles

In this case, one of the excircles lies along the side of the triangle creating an infinite number of extouch points. In such a case there is not really viable meaning of a Nagel point.

Section 6.13: Fermat Points

The Fermat point of a triangle is the point within the triangle or on its edge at which the sum of the distances to the vertices is a minimum. In Euclidean geometry, for triangles with maximum angle measure less than 120°, equilateral triangles can be drawn at each side of the triangle and the vertices opposite each side of the triangle are connected. The intersection of these three lines turns out to be the Fermat point of the triangle. For triangles with maximum angle measure greater than 120° the Fermat point is the vertex of the obtuse angle.

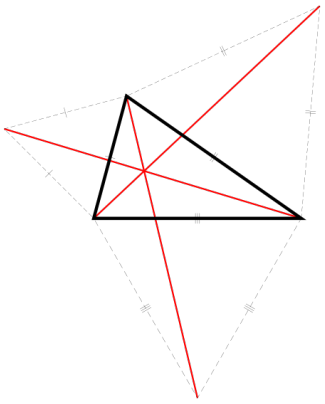


Figure 6.13.1 Constructing the Fermat point of a Euclidean triangle

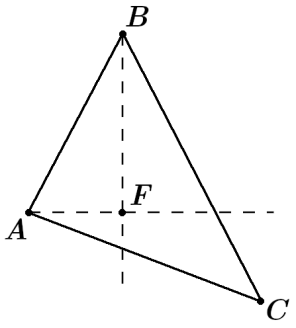


Figure 6.13.2 Finding the Fermat point for a taxicab triangle with no angles compassing a properly oriented right angle

Consider the triangle in Figure 6.13.2 formed by $A = (x_a, y_a)$, $B = (x_b, y_b)$, and $C = (x_c, y_c)$ where no angle fully encompasses a properly oriented right angle (Definition 6.7.1). Then one angle of the triangle will straddle a horizontal line (choose $\angle A$) and one angle will straddle a vertical line (choose $\angle B$). The sum of the distances from point $P = (x, y)$ to the vertices is

$$d = |x - x_a| + |y - y_a| + |x - x_b| + |y - y_b| + |x - x_c| + |y - y_c|$$

To handle the absolute values, divide the triangle into regions using the horizontal and vertical lines through the two vertices. In each region the absolute value can be resolved into either $x - x_*$ or $x_* - x$, etc. To find candidates for the minimum of this function in each region, the partial derivatives with respect to x and y are taken, set equal to zero, and solved. In this case, the partial derivatives will always be a constant of ± 1 or ± 2. So, no minimum candidates reside within the interior of the regions. We therefore must check the boundaries of the regions, so the sides of the triangles and the horizontal and vertical lines splitting the triangle.

Case 1: $y = y_a$ to the left of $x = x_b$. Then $d = x - x_a + 0 + x_b - x + y_b - y_a + x_c - x + y_a - y_c$ which simplifies to $d = -x - x_a + x_b + y_b + x_c - y_c$. This expression is minimized for $x \in (x_a, x_b)$ by selecting the largest value in the interval, x_b, giving $d = -x_a + y_b + x_c - y_c$ at point (x_b, y_a), the intersection of the region boundaries.

Case 2: $y = y_a$ to the right of $x = x_b$. Then $d = x - x_a + 0 + x - x_b + y_b - y_a + x_c - x + y_a - y_c$ which simplifies to $d = x - x_a - x_b + y_b + x_c - y_c$. This expression is minimized for x from x_b to the edge of the triangle by selecting the smallest value in the interval, x_a, again giving $d = -x_a + y_b + x_c - y_c$ at point (x_b, y_a), the intersection of the region boundaries.

Case 3: $x = x_b$ above $y = y_a$. Then $d = x_b - x_a + y - y_a + 0 + y_b - y + x_c - x_b + y - y_c$ which simplifies to $d = y - x_a - y_a + y_b + x_c - y_c$. This expression is minimized for $y \in (y_a, y_b)$ by selecting the smallest value in the interval, y_a, again giving $d = -x_a + y_b + x_c - y_c$ at (x_b, y_a).

Case 4: $x = x_b$ below $y = y_a$. Then $d = x_b - x_a + y_a - y + 0 + y_b - y + x_c - x_b + y - y_c$ which simplifies to $d = -y - x_a + y_a + y_b + x_c - y_c$. This expression is minimized for y from the edge of the triangle to y_a by selecting the largest value in the interval, y_a, again giving $d = -x_a + y_b + x_c - y_c$ at point (x_b, y_a).

Case 5: Along $\overline{AB}$. Resolving the absolute value terms gives

$$d = (-x_a + y_b + x_c - y_c) + (x_b - x) + (y - y_a)$$

The first term is the minimum found in the previous cases and the last two terms are always non-negative on $\overline{AB}$ and never both zero. So no matching or smaller minimum occurs on $\overline{AB}$.

Case 6: Along $\overline{BC}$ above $y = y_a$. Resolving the absolute value terms gives

$$d = (-x_a + y_b + x_c - y_c) + (x - x_b) + (y - y_a)$$

The first term is the minimum found in the previous cases and the last two terms are always non-negative on $\overline{BC}$ above $y = y_a$ and never both zero. So no matching or smaller minimum occurs on $\overline{BC}$ above $y = y_a$.

Case 7: Along $\overline{BC}$ below $y = y_a$. Resolving the absolute value terms gives

$$d = (-x_a + y_b + x_c - y_c) + (x - x_b) + (y_a - y)$$

The first term is the minimum found in the previous cases and the last two terms are always non-negative on $\overline{BC}$ below $y = y_a$ and never both zero. So no matching or smaller minimum occurs on $\overline{BC}$ below $y = y_a$.

Case 8: Along $\overline{AC}$ right of $x = x_b$. Resolving the absolute value terms gives

$$d = (-x_a + y_b + x_c - y_c) + (x - x_b) + (y_a - y)$$

The first term is the minimum found previously and the last two terms are always non-negative on $\overline{AC}$ below $y = y_a$ and never both zero. So no matching or smaller minimum occurs on $\overline{AC}$ below $y = y_a$.

Case 9: Along $\overline{AC}$ left of $x = x_b$. Resolving the absolute value terms gives

$$d = (-x_a + y_b + x_c - y_c) + (x_b - x) + (y_a - y)$$

The first term is the minimum found in the previous cases and the last two terms are always non-negative on $\overline{AC}$ above $y = y_a$ and never both zero. So no matching or smaller minimum occurs on $\overline{AC}$ above $y = y_a$.

Theorem 6.13.1 *The Fermat point for the taxicab triangle formed by $A = (x_a, y_a)$, $B = (x_b, y_b)$, and $C = (x_c, y_c)$ with $\angle A$ straddling or coinciding with a horizontal line and $\angle B$ straddling or coinciding with a vertical line is (x_b, y_a).*

This result should not be surprising. Taxicab distances are minimized when a line segment is oriented parallel to a coordinate axis. By selecting a point that is horizontal from one vertex and vertical from another, two of the point-to-vertex distances are being minimized, which is the best that can be done.

Unlike Euclidean geometry, there does not appear to be a specific angle size for the triangle at which this analysis degenerates. However, we must still consider the case where one angle of the triangle is large enough and oriented in such a way as to encompass a properly oriented right triangle as in Figure 6.13.3 In this case the horizontal and vertical lines that divide the triangle pass through a single vertex. A very similar analysis to the one above will follow with the result that, like Euclidean case of a very obtuse angle, the Fermat point is the vertex of the obtuse angle.

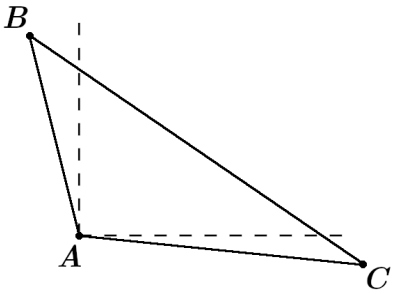


Figure 6.13.3 Finding the Fermat point (vertex A) for a triangle with one angle encompassing a properly oriented right angle

Theorem 6.13.2 *The Fermat point for a taxicab* $\triangle ABC$ *with* $\angle A$ *encompassing a properly oriented right angle is the vertex* A*.*

Section 6.14: The Erdös–Mordell Theorem

In Euclidean geometry the Erdős–Mordell Theorem states that if a point P lies inside a triangle then the sum of the distances from the point to the vertices is at least twice the sum of the distances from the point to the sides (along perpendicular bisectors). This was investigated in taxicab geometry in a 2009 research paper by Nilgün Sönmez entitled "A Taxicab Version of the Erdös–Mordell Theorem," but curiously the research retained the *Euclidean* concept of distance from a point to a line. As we saw in Section 2.3, this concept is very different in taxicab geometry. In this section we will attempt to tweak the research of Sönmez and examine this theorem in *pure taxicab geometry.*

In some of what follows reference will be made to a "properly oriented" right angle or triangle, a concept which was introduced in Section 6.7. It simply means the rays of the right angle are parallel to the coordinate axes.

Lemma 6.14.1 *Let* $\triangle ABC$ *be an isosceles and properly oriented right triangle;* P *a point in the interior of the triangle;* d_A, d_B, *and* d_C *the taxicab distances from* P *to the vertices of the triangle; and,* d_a, d_b, *and* d_c *the taxicab distances to the sides of the triangle. Then*

$$d_A + d_B + d_C > 2(d_a + d_b + d_c)$$

PROOF: Using Figure 6.14.1 and without loss of generality, place the triangle in the first quadrant with the right $\angle C$ at the origin and point $P = (x, y)$. Since the triangle is isosceles $a = b$ and we have

$$\begin{aligned} d_A &= a - y + x & \qquad d_a &= y \\ d_B &= a - x + y & \qquad d_b &= x \\ d_C &= x + y & \qquad d_c &= a - y - x \end{aligned}$$

Note here that we have used the taxicab concept of distance to a line in determining d_a, d_b, and d_c. In particular, since the hypotenuse is a perfect diagonal, either a horizontal or vertical path d_c can be taken from the point to the line. Therefore,

$$d_A + d_B + d_C = x + y + 2a > 2a = 2(d_a + d_b + d_c)$$ □

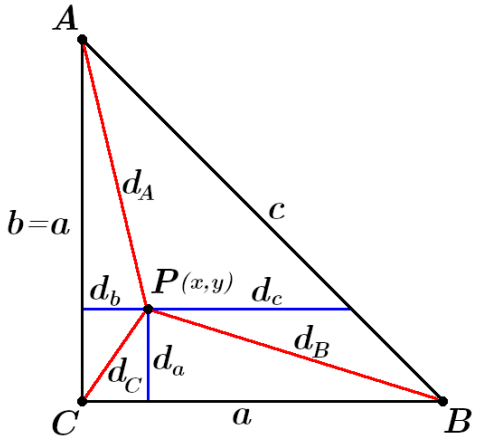


Figure 6.14.1 The Erdös–Mordell Theorem for a properly oriented isosceles right triangle

Lemma 6.14.2 *Let $\triangle ABC$ be a properly oriented right triangle; P a point in the interior of the triangle; d_A, d_B, and d_C the taxicab distances from P to the vertices of the triangle; and, d_a, d_b, and d_c the taxicab distances to the sides of the triangle. Then*

$$d_A + d_B + d_C > 2(d_a + d_b + d_c)$$

PROOF: Without loss of generality, place the triangle in the first quadrant with the right $\angle C$ at the origin and point $P = (x, y)$, and assume that $b > a$ (Figure 6.14.2). Since d_c is the shortest path to $\overline{AB}$, $d_c < d'_c$ which means $d_a + d_c < d_a + d'_c < b$. In addition, $d_b + d_c < a$. Therefore,

$$\begin{aligned} d_a + d_b + 2d_c &< a + b \\ 2d_a + 2d_b + 2d_c &< d_b + b + d_a + a \\ 2(d_a + d_b + d_c) &< (d_b + b - d_a) + (d_a + a - d_b) + (d_a + d_b) \\ 2(d_a + d_b + d_c) &< d_A + d_B + d_C \end{aligned}$$

□

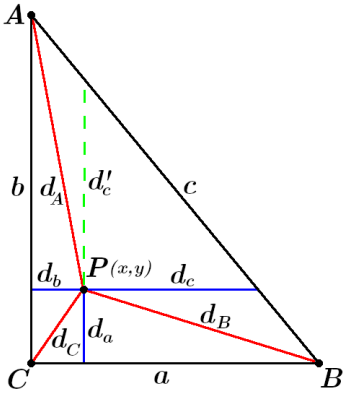


Figure 6.14.2 The Erdös–Mordell Theorem for a properly oriented right triangle

In 2019 Petrovic, Malesevic, and Banjac refined the work of Sönmez. They correctly find the taxicab distance from a point to a line and begin with a counterexample to Sönmez's conclusions (Figure 6.14.3). So, while we have made progress, it seems this is as far we can go if we want to keep the factor of 2 between the sums.

$$\begin{aligned} A &= (0, 30) & B &= (-20, 0) \\ C &= (40, 0) & P &= (0, 2) \\ d_A &= y_a - y_p = 28 & d_a &= y_p = 2 \\ d_B &= -x_b + y_p = 22 & d_b &= y_a - y_p = 28 \\ d_C &= x_c + y_p = 42 & d_c &= -\left(x_b - \frac{x_b y_p}{y_a}\right) = \frac{56}{3} \end{aligned}$$

$$d_A + d_B + d_C = 92 \not\geq 97.\overline{3} = 2(d_a + d_b + d_c)$$

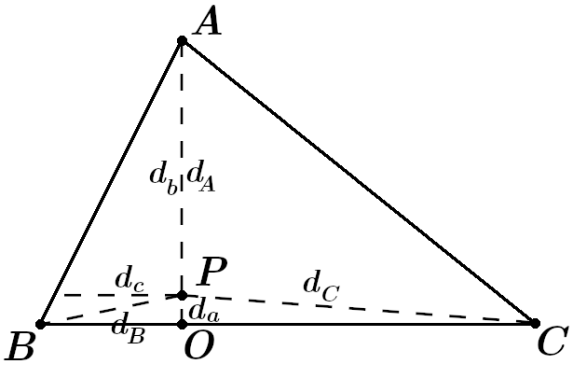


Figure 6.14.3 A counterexample to the conclusions of Sönmez's taxicab version of the Erdös-Mordell Theorem

In their research Petrovic, et. al show that a factor of 1.5 suffices in taxicab geometry for a triangle with one side parallel to a coordinate axis. The result for a general triangle is unproven.

Theorem 6.14.3 *Let* $\triangle ABC$ *be a triangle with one side parallel to a coordinate axis;* P *a point in the interior of the triangle;* d_A, d_B, *and* d_C *the taxicab distances from* P *to the vertices of the triangle; and,* d_a, d_b, *and* d_c *the taxicab distances to the sides of the triangle. Then*

$$d_A + d_B + d_C > \frac{3}{2}(d_a + d_b + d_c)$$

Chapter 7
Conic Sections

Section 7.1: Taxicab Cones and Conic Sections

In Euclidean geometry, the topic of conics (circles, ellipses, parabolas, and hyperbolas) is a large and complex one. There are multiple ways of defining each of the conics in Euclidean geometry, but they all end up describing the same two-dimensional curve. Life is not so simple in taxicab geometry.

Before we delve into taxicab conics in detail, let us review the different formulations of each conic in Euclidean geometry along with other terminology.

Conic Definitions with Two Foci

Most of the conics can be defined algebraically with respect to two fixed points called foci. An ellipse is the set of points for which the sum of the distances to the foci is constant. When the foci are not distinct, the ellipse is a circle. A hyperbola is the set of points for which the *difference* of the distances to the foci is constant. A parabola does not have a two-foci definition.

Conic Definitions with a Focus and a Directrix

Most of the conics have an alternate algebraic definition involving a fixed point (still called a focus) and line called the directrix. The most natural and common of these is for a parabola: the set of points for which the distance to the focus equals the distance to a line. Alternatively: a parabola is the set of points where the ratio of [the distance to the focus] to [the distance to the directrix] is one. If we focus on just one of the foci, an ellipse is the set of points such that the ratio of [the distance to the focus] to [the distance to the directrix] is a constant less the one. So, a non-circular ellipse has two directrices, one for each focus. Finally, again considering one focus at a time, a hyperbola is the set of points such that the ratio of [the distance to the focus] to [the distance to the directrix] is a constant greater the one.

Although a circle does not have a natural definition in terms of a directrix, it could be thought of as an ellipse with a directrix "at infinity."

Eccentricity

The focus-directrix definitions for the conics separate the conics based on the value of the ratio of [distance to focus] to [distance to directrix]. This ratio is call eccentricity. Since a circle does not really have a directrix, or its directrix is "at infinity," its eccentricity is considered to be zero.

Conics as Cross Sections of a Cone

All of the conics can alternately be defined as a curve obtained from the intersection of the surface of a right circular cone with a plane. By changing the angle of the plane, different kinds of curves are formed by the intersection. If the plane is horizontal to the base of the cone, the intersection is a circle. If the plane is slightly angled, the intersection is an ellipse. If the angle of the plane is greater than the angle of the "side" of the cone, a parabola is formed. A hyperbola is formed by intersecting the plane perpendicular to the bases of a double cone - two cones stacked on their apexes.

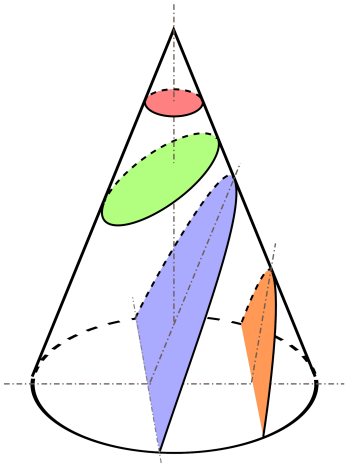


Figure 7.1.1 Euclidean conic sections formed from the surface of a right circular cone and intersecting planes (outlines of the shaded regions)

Note that in Euclidean geometry, the curves formed in the intersecting plane are themselves the actual conics. No projection down to the xy-plane is necessary.

What to Expect for Conics in Taxicab Geometry

Since circles in taxicab geometry have a square shape, a right circular cone in taxicab geometry will look like Figure 7.1.1, a figure which is typically called a right pyramid in Euclidean geometry. Therefore, intersections of a plane and a taxicab right circular cone will take on very different shapes than in Euclidean geometry.

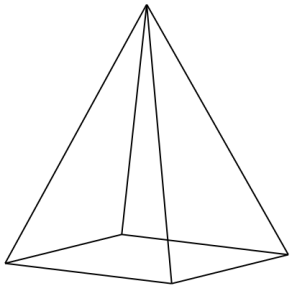


Figure 7.1.2 A taxicab right circular cone

[As a side note, one way of visualizing a taxicab right circular cone is to imagine revolving a line from the origin to a point in the first quadrant around the x-axis or y-axis. This produces a solid of revolution in the shape of a cone. In Section 8.4 we examine how to do this properly in taxicab geometry. But, the result is what is shown in Figure 7.1.2.]

In terms of the algebraic definitions of conics, the two formulations themselves will generally ***not*** agree in taxicab geometry to the point of often producing non-overlapping sets of figures. However, taxicab conical sections will turn out to agree with the focus-directrix formulations *provided* the intersections are projected down to the xy-plane. This was first examined by Richard Laatsch in 1982.

Section 7.2: Circles

Since the shape of a taxicab circle is well known to us by now, we begin by jumping straight to the experiment of intersecting a taxicab right circular cone with a plane that is parallel to the base of the cone. The result is a shape like that in Figure 7.2.1. So, it certainly appears that taxicab circles are indeed conic sections.

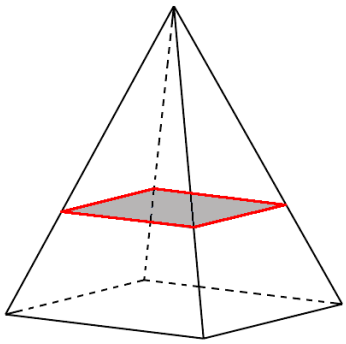


Figure 7.2.1 A circle (red) as a conic section in taxicab geometry

As a side note, it is easy to view taxicab right circular cones as what we would call a Euclidean pyramid. But, if we choose to do this, we must be careful about the orientation of the pyramid with respect to the axes. Circles only have a "diamond" shape in taxicab geometry, with the vertices on (lines parallel to) the x-axis and y-axis. Other orientations of the Euclidean pyramids will not produce a base with a taxicab circle and will therefore not be a taxicab circular cone.

As demonstrated in Section 2.2, the algebraic definition of a taxicab circle is the set of points a fixed distance r, called the radius, from a center point (h, k), which leads to an equation of the form

$$|x - h| + |y - k| = r \tag{2.2.1}$$

When we consider ellipses in Section 7.3, we will discover that the circle is a special case of the two-focus algebraic definition as it is in Euclidean geometry.

Section 7.3: Ellipses

Two-Foci Definition

In the two-foci definition, an ellipse is the set of all points that are equidistant from two fixed points called foci, or equivalently the set of points $P = (x, y)$ such that the sum of the distances from P to the foci $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$ is constant:

$$d(P, F_1) + d(P, F_2) = s$$

$$|x - x_1| + |y - y_1| + |x - x_2| + |y - y_2| = s \tag{7.3.1}$$

Several cases result from this equation. In some instances the value of s relative to distance between the foci affects the shape of the conic. For consistency, let $d_f = d(F_1, F_2)$ represent the distance between the foci.

Case 1: $d_f = 0$. Then $F_1 = F_2 = (h, k)$ and the ellipse equation reduces to

$$|x - h| + |y - k| = \frac{s}{2}$$

which is a taxicab circle of radius $s/2$ centered at (h, k), a version of which is shown in Figure 7.3.1c. The interpretation here is: non-distinct foci are equivalent to a horizontal slicing plane. These figures correspond to the circles described in Section 7.2.

Case 2: $0 < d_f < s$ with F_1 and F_2 lying on a horizontal or vertical line (so that either $y_1 = y_2$ or $x_1 = x_2$). Taking the horizontal case for illustration, the ellipse equation reduces to

$$|x - x_1| + |x - x_2| + 2|y - k| = s$$

If we graph the ranges of x values and y values separately to resolve the absolute value operations, we get 6 lines to graph. The result is a hexagon akin to the ellipse in Figure 7.3.1b.

Case 3: $0 < d_f < s$ with F_1 and F_2 not lying on a horizontal or vertical line. The ellipse equation does not simplify and 8 ranges of x values and y values must be graphed separately. This yields an octagon as in Figure 7.3.1a.

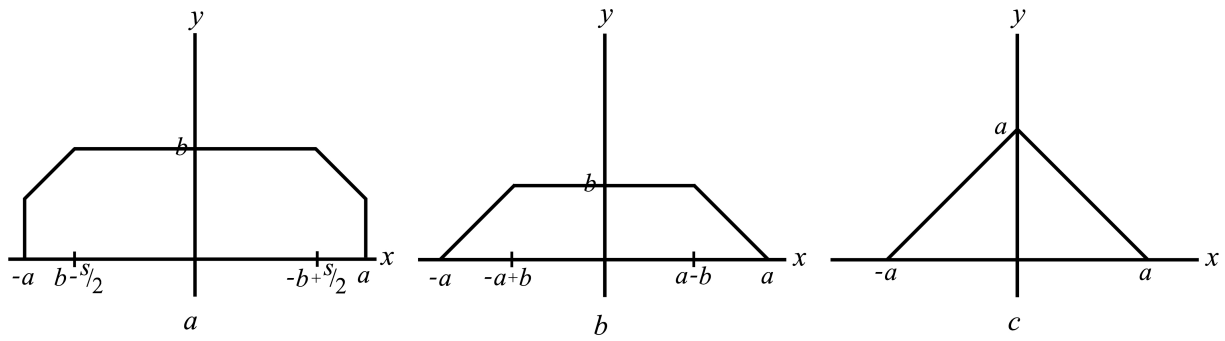


Figure 7.3.1 Non-degenerate, two-foci taxicab ellipses. Shown are the upper halves of a) the octagonal ellipse, b) the hexagonal ellipse, and c) the circular ellipse (a taxicab circle).

These cases for non-degenerate ("true") two-foci taxicab ellipses taking the forms of taxicab circles, hexagons, and octagons can be described by a piecewise function. If we consider a taxicab ellipse with major axis length $2a$, minor axis length $2b$, and s the sum of the distances from a point on the ellipse to the foci, the function

$$f(x) = \begin{cases} x + \frac{s}{2} & -a \le x < b - \frac{s}{2} \\ b & b - \frac{s}{2} \le x < -b + \frac{s}{2} \\ -x + \frac{s}{2} & -b + \frac{s}{2} \le x \le a \end{cases} \tag{7.3.2}$$

describes the upper half of an ellipse centered at the origin. This function will generally cover the sphere ($s = 2a$ and $a = b$, i.e. $d_f = 0$), hexagon ($s = 2a$ and $a > b$, i.e. foci on horizontal or vertical line), and octagon ($s > 2a$) cases for a two-foci taxicab ellipse, with the addition of vertical lines for the left and right sides of the octagon ellipse.

The other cases not addressed here where $d_f = s$ lead to degenerate forms of ellipses such as line segments and rectangular regions.

The Focus-Directrix Definition

An ellipse is the set of points such that the ratio of [the distance to the focus] to [the distance to the directrix] is a constant e less the one. Using Theorem 2.3.1 for the distance from a point to a line of the form $ax + by + c = 0$

$$\frac{d(P,F)}{d(P,l)} = e < 1$$

$$|x - h| + |y - k| = e \cdot \frac{|ax + by + c|}{\max\{|a|, |b|\}} \tag{7.3.3}$$

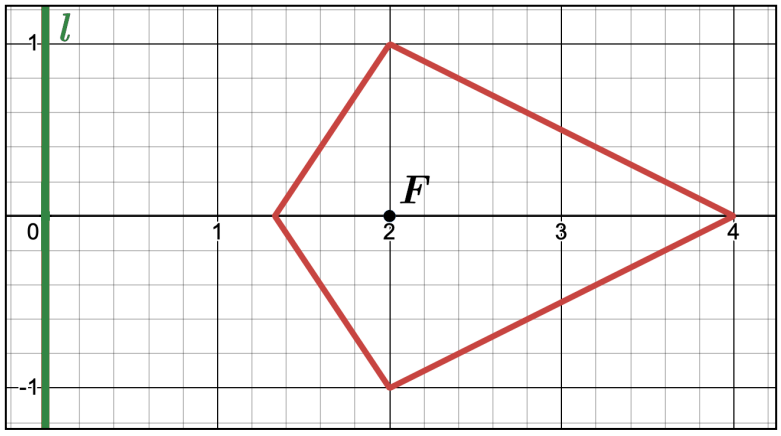


Figure 7.3.2 A focus-directrix taxicab ellipse with vertical directrix ($F = (2, 0)$, **directrix**: $x = 0$, and $e = 0.5$) defined by:

$$|x - 2| + |y| = \frac{1}{2}|x|$$

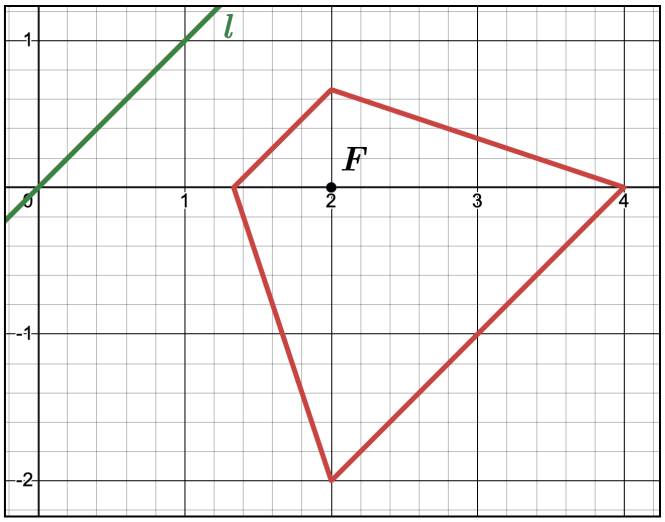


Figure 7.3.3 A focus-directrix taxicab ellipse with diagonal directrix ($F = (2, 0)$, **directrix**: $y = x$, and $e = 0.5$) defined by:

$$|x - 2| + |y| = \frac{1}{2}|x - y|$$

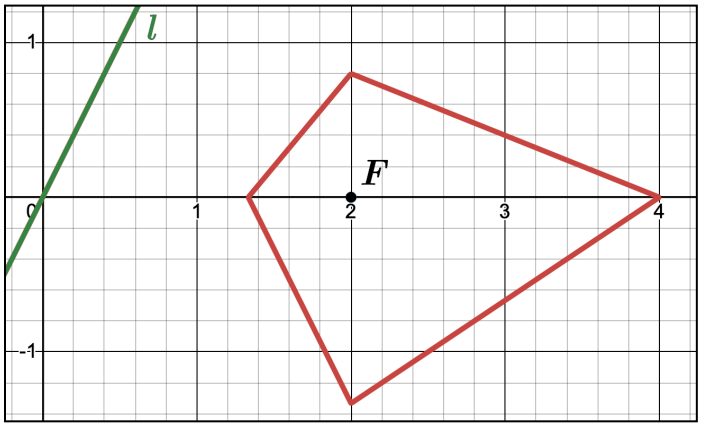


Figure 7.3.4 A focus-directrix taxicab ellipse with arbitrary directrix ($F = (2, 0)$, **directrix**: $y = 2x$, and $e = 0.5$) defined by:

$$|x - 2| + |y| = \frac{1}{2}\frac{|2x - y|}{2}$$

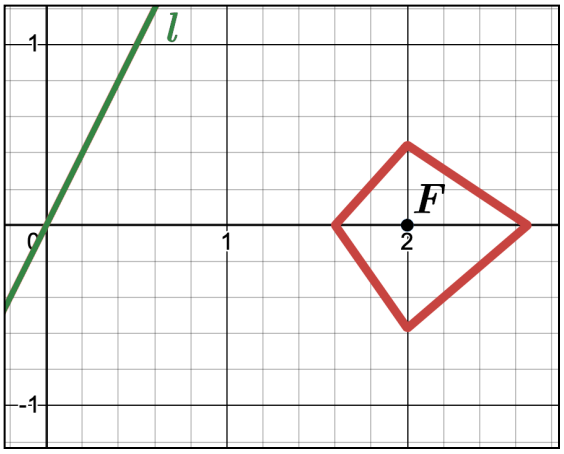


Figure 7.3.5 The focus-directrix taxicab ellipse of Figure 7.3.4 with a smaller eccentricity ($F = (2, 0)$, **directrix**: $y = 2x$, and $e = 0.25$) defined by:

$$|x - 2| + |y| = \frac{1}{4}\frac{|2x - y|}{2}$$

From these examples that cover the various orientations of the directrix, focus-directrix ellipses are always quadrilaterals.

As with Euclidean ellipses, as eccentricity is reduced towards zero a taxicab ellipse become more circular, and as eccentricity approaches one the ellipse becomes more elongated.

To wrap up this subsection, we give the piecewise function definition for the upper half of the (symmetric) focus-directrix ellipse with eccentricity $e < 1$, vertical directrix $x = -c$, and focus $F = (0, 0)$ as we will need it when investigating a "true" taxicab ellipsoid formed from a surface or solid of revolution in Chapter 8:

$$f(x) = \begin{cases} (e+1)x + e \cdot c, & \frac{-ec}{e+1} \leq x < 0 \\ (e-1)x + e \cdot c, & 0 \leq x \leq \frac{-ec}{e-1} \end{cases} \tag{7.3.4}$$

Ellipses as Conic Sections

The slicing of a taxicab right circular cone by a shallow plane produces a quadrilateral figure on the surface of the cone (see Figure 7.3.6). This figure resembles those seen in the focus-directrix definition of an ellipse, and indeed, if this curve is projected onto the xy-plane the resulting figure is a focus-directrix ellipse.

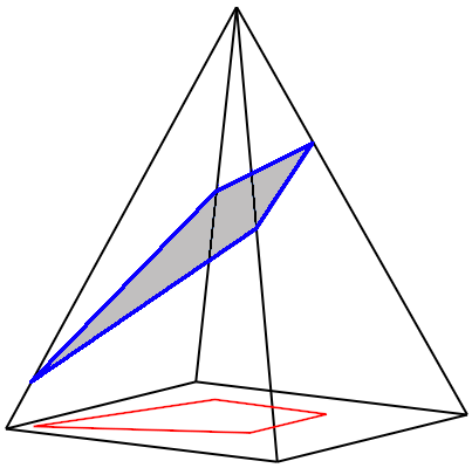


Figure 7.3.6 A taxicab ellipse (red) as a projection of a conical section (blue).

Non-trivial two-foci taxicab ellipses cannot be derived from conical sections as there is no way to intersect a taxicab right circular cone with a plane that produces a hexagonal or octagonal shape on the surface of the cone. Therefore, two-foci "ellipses" appear to be a distinct concept and construct in taxicab geometry independent of focus-directrix ellipses and their associated conical sections. This is in stark contrast to Euclidean geometry where all three definitions are equivalent.

Taking a specific example, consider the taxicab right circular cone $|x-2|+|y|=\frac{1}{4}|z|$ intersected by the plane $z=-2x+y$ (Figure 7.3.7). The shape formed on the surface of the cone (blue) is not a taxicab focus-directrix ellipse. But, if we project this curve onto the xy-plane by eliminating the z variable, the equation of a taxicab focus-directrix ellipse appears:

$$|x-2|+|y|=\frac{1}{2}\frac{|-2x+y|}{2}$$

Viewing the cone from above depicts the curve projected onto the xy-plane (Figure 7.3.8), a projection that matches the focus-directrix ellipse in Figure 7.3.4.

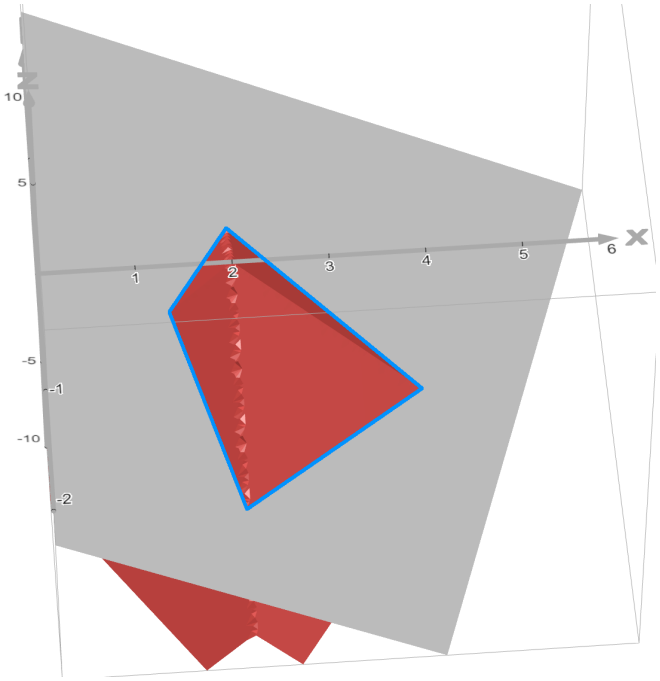


Figure 7.3.7 Intersection of the taxicab right circular cone $|x-2|+|y|=\frac{1}{4}|z|$ with the plane $z=-2x+y$ resulting in a shape that when projected onto the xy-plane forms a taxicab focus-directrix ellipse

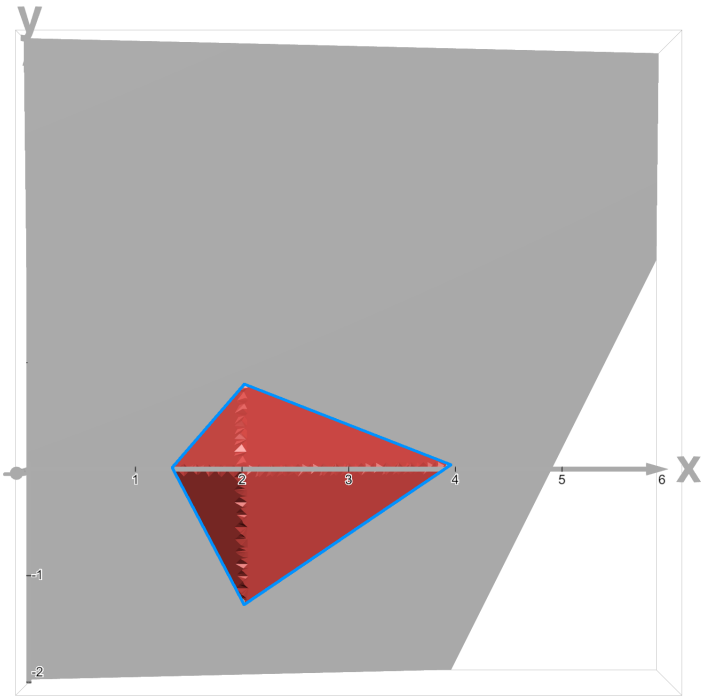


Figure 7.3.8 The projection of the curve on the taxicab cone in Figure 7.3.7 to the xy-plane as seen from above the pyramid. This is the same focus-directrix ellipse as the one pictured in Figure 7.3.4.

Theorem 7.3.1 The length of the horizontal axis of a focus-directrix ellipse with eccentricity e, steep directrix, and focus to directrix distance h is

$$a_h = \frac{2eh}{1-e^2}$$

Theorem 7.3.2 The length of the vertical axis of a focus-directrix ellipse with eccentricity e, steep directrix with m, and focus to directrix distance h is

$$a_v = \frac{2ehm^2}{m^2-e^2}$$

Theorem 7.3.3 The area of a focus-directrix ellipse with eccentricity e, steep directrix with slope m, and focus to directrix distance h is

$$A = \frac{2e^2h^2m^2}{(1-e^2)(m^2-e^2)}$$

PROOF: Since the ellipse is composed of properly oriented triangles, they are contained in an enclosing rectangle that is properly oriented. As such, the taxicab length and width of the rectangle equal the Euclidean lengths and its area can be computed in the normal Euclidean manner. The length and width of this rectangle are simply the lengths of the horizontal and vertical axes of the ellipse. Therefore,

$$\begin{aligned} A &= \frac{1}{2}a_h a_v \\ &= \frac{1}{2}\frac{2eh}{1-e^2}\frac{2ehm^2}{m^2-e^2} \\ &= \frac{2e^2h^2m^2}{(1-e^2)(m^2-e^2)} \end{aligned}$$

□

Theorem 7.3.4 The area of a focus-directrix ellipse with eccentricity e, vertical directrix, and focus to directrix distance h is

$$A = \frac{2e^2h^2}{1-e^2}$$

PROOF: Divide through the formula given in Theorem 7.3.3 by m^2 and let $m \to \infty$. □

Section 7.4: Parabolas

The only algebraic definition for a parabola is the focus-directrix definition. A parabola is a curve in which each point $P = (x, y)$ on the curve is equidistant from a fixed point called a focus $F = (h, k)$ and a straight line called a directrix l, or, equivalently, the ratio of these distances (the eccentricity) is one. Using Theorem 2.3.1 for the distance from a point to a line of the form $ax + by + c = 0$, we have

$$d(P, F) = d(P, l)$$

$$|x - h| + |y - k| = \frac{|ax + by + c|}{\max\{|a|, |b|\}} \tag{7.4.1}$$

As described in Section 2.3, the shortest path to a line is dependent on the slope of the line - a horizontal path for a steep line and a vertical path for a shallow line. This is the why the max{} function appears in the formulation of the parabola.

As with ellipses, a number of cases must be examined.

Case 1: The directrix has shallow slope. Then $|a| < |b|$ and we have

$$|x - h| + |y - k| = \frac{|ax + by + c|}{|b|} = \left|\frac{a}{b}x + y + \frac{c}{b}\right|$$

Figure 7.4.1 shows a shallow slope directrix parabola. The sub-case where the directrix is horizontal produces the most Euclidean-like taxicab parabola: a figure opening upward and symmetric about an axis of symmetry through the vertex and perpendicular to the directrix (akin to the vertical directrix case shown in Figure 7.4.4). When the directrix is not horizontal, the parabola is not symmetric about a line through the vertex.

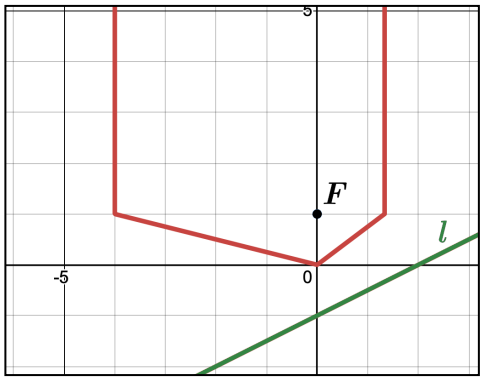


Figure 7.4.1a A shallow directrix parabola ($F = (0, 1)$, **directrix**: $y = x/2 - 1$):

$$|x| + |y - 1| = \frac{|-x + 2y + 2|}{2}$$

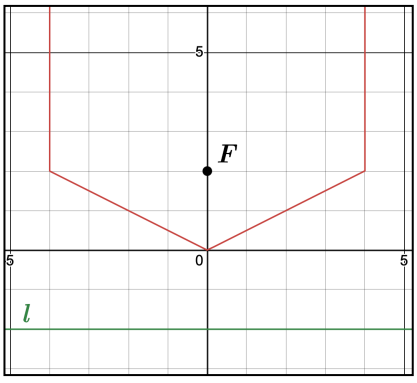


Figure 7.4.1b A horizontal directrix parabola ($F = (0, 2)$, **directrix**: $y = -2$):

$$|x| + |y - 2| = |y + 2|$$

Case 2: The directrix has slope $m = \pm 1$. Then $|a| = |b|$ and we have

$$|x-h| + |y-k| = \left|x + y + \frac{c}{b}\right|$$

The resulting parabola will be similar to Figure 7.4.2. Symmetry has returned in this case.

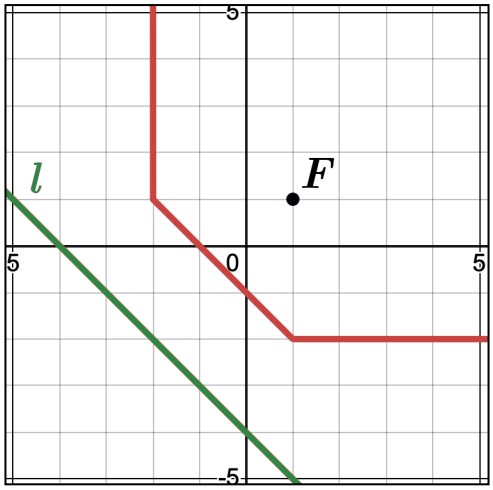


Figure 7.4.2 A diagonal directrix parabola ($F = (1,1)$, **directrix**: $y = -x - 4$) defined by $|x-1| + |y-1| = \left|x + y + 4\right|$

Case 3: The directrix has steep slope. Then $|a| > |b|$ and this will mimic Case 1 but the parabola will open horizontally rather than vertically (Figure 7.4.3).

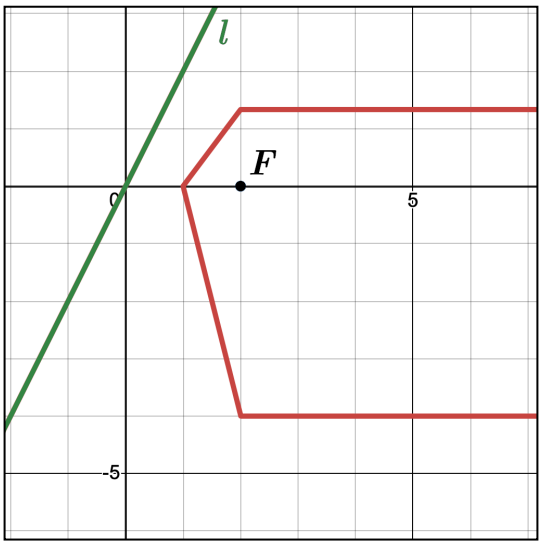


Figure 7.4.3 A steep directrix parabola ($F = (2,0)$, **directrix**: $y = 2x$) defined by:

$$|x-2| + |y| = \left|x - \frac{y}{2}\right|$$

Case 4: The directrix is vertical. This mimics the symmetric sub-case of Case 1 produced by a horizontal directrix, but now the parabola opens horizontally (Figure 7.4.4).

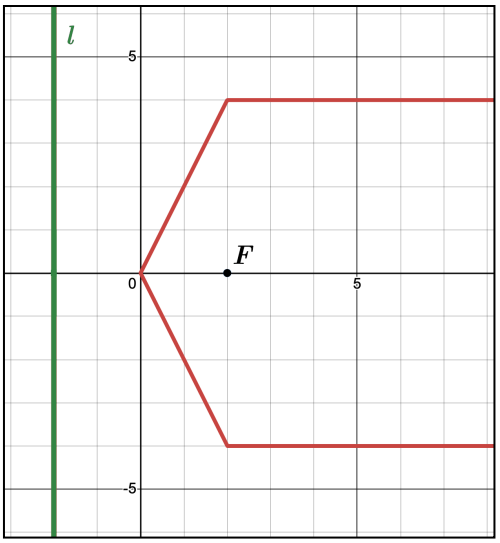


Figure 7.4.4 A vertical directrix parabola ($F = (2, 0)$, **directrix**: $x = -2$) defined by $|x - 2| + |y| = |x + 2|$

Parabolas as Conic Sections

Taxicab parabolas can be obtained by intersecting the surface of a taxicab right circular cone with a steep plane. But, as mentioned in Section 7.1, the curve formed in the plane / on the surface of the cone itself does not match the algebraic formulation of taxicab parabolas above. The curve must be projected onto the xy-plane in order to be a taxicab parabola (Figure 7.4.5). This is in contrast to Euclidean geometry where such a projection is not necessary.

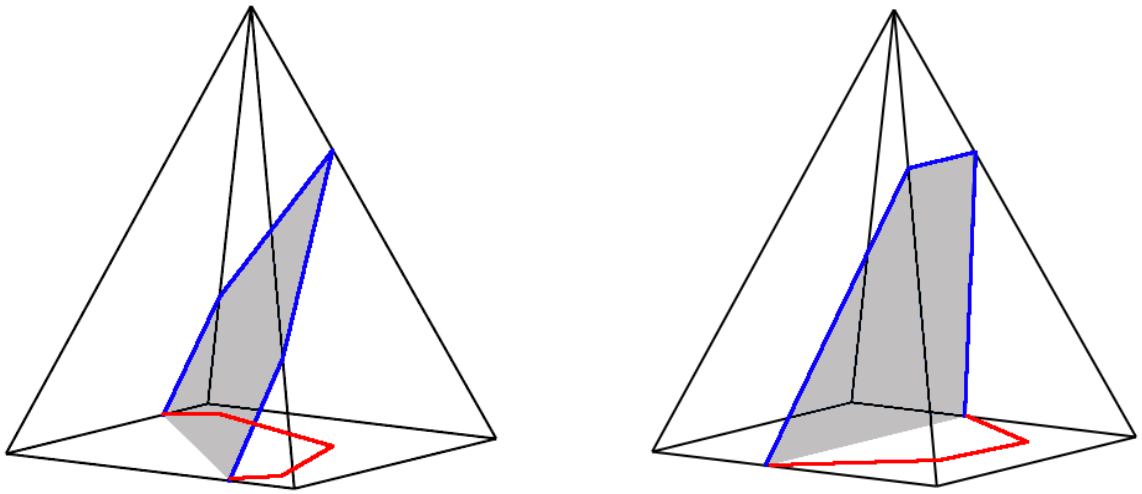


Figure 7.4.5 Taxicab parabolas (red) as projections of conical sections (blue). The left corresponds to a parabola like that in Figure 7.4.4 whereas the right to Figure 7.4.2. Rotating the intersecting plane between the positions shown in these two examples will produce asymmetrical parabolas like Figures 7.4.1 and 7.4.3.

Taking a specific example, consider the taxicab right circular cone defined by $|x|+|y|=\frac{1}{2}|z|$ and intersect it with the the plane $z=2x-y-6$ (Figure 7.4.6). The shape formed on the surface of the cone (blue) is not a taxicab parabola. But, if we project this curve onto the xy-plane by eliminating the z variable, the equation of a taxicab parabola appears:

$$|x|+|y|=\frac{|2x-y-6|}{2}$$

This parabola is shown in detail in Figure 7.4.7b and from above the cone in blue in Figure 7.4.7a.

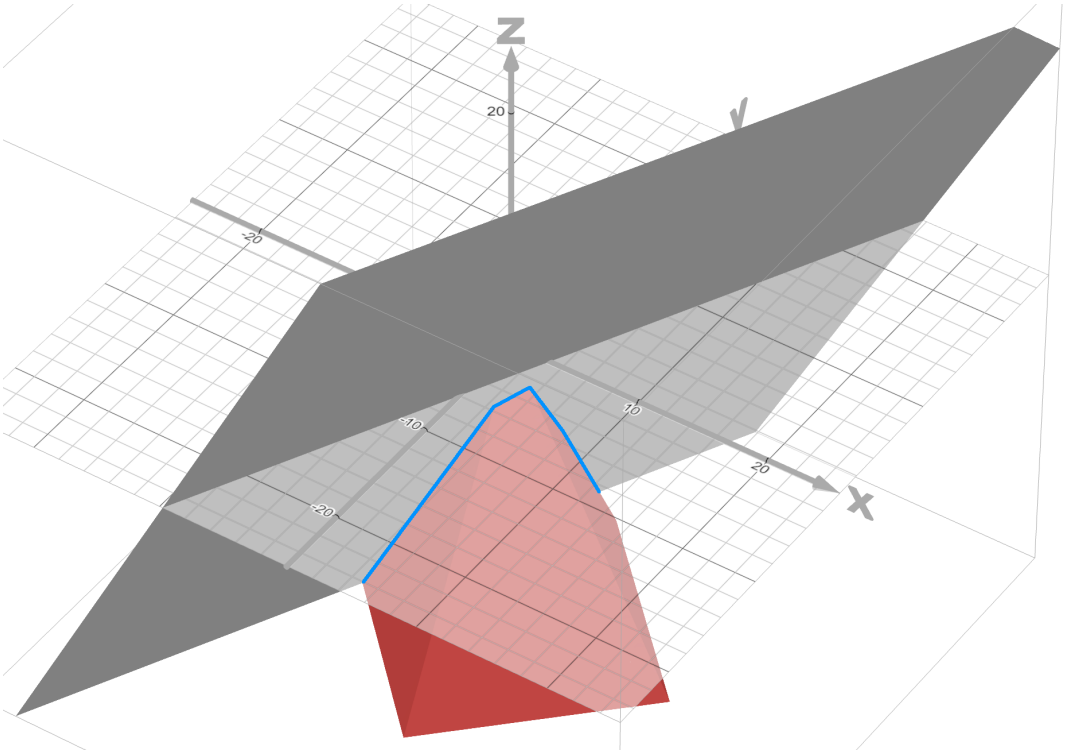


Figure 7.4.6 Intersection of the taxicab right circular cone $|x|+|y|=\frac{1}{2}|z|$ with the plane $z=2x-y-6$ resulting in a shape that when projected onto the xy-plane forms a taxicab parabola.

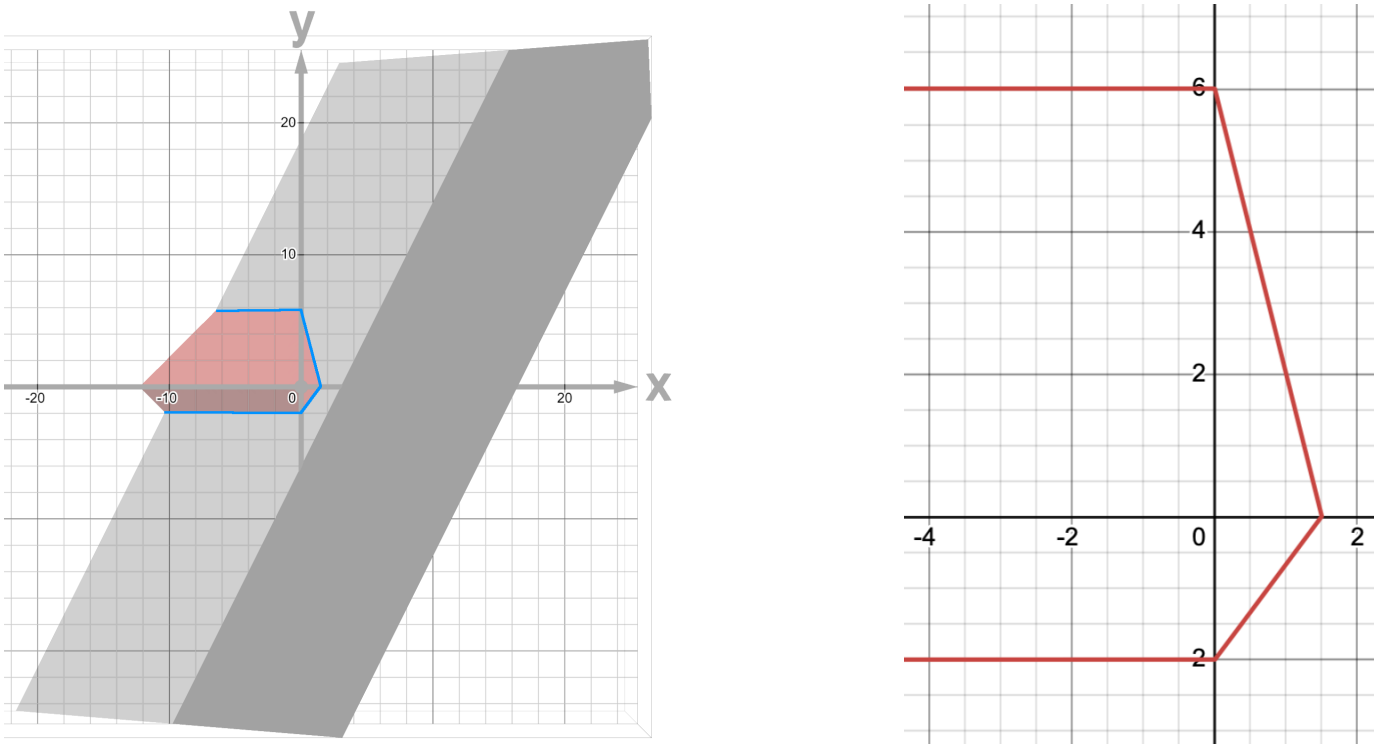


Figure 7.4.7 The projection of the curve on the taxicab cone to the xy-plane a) as seen from above the pyramid, and b) graphed from its algebraic equation.

Piecewise Function Definition for a Parabola

As a final note for this subsection, the equation for upper half of an rightward-facing symmetric parabola similar to the one pictured in Figure 7.4.4 with focus $F = (k, 0)$ and directrix $x = -k$ has a piecewise function definition of

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq k \\ 2k, & x > k \end{cases} \tag{7.4.2}$$

Section 7.5: Hyperbolas

Two-Foci Definition

A hyperbola can be described as set of points $P = (x, y)$ such that the *absolute difference* of the distances from P to the foci $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$ is constant:

$$d(P, F_1) - d(P, F_2) = s$$

$$\Big|\, |x - x_1| + |y - y_1| - |x - x_2| - |y - y_2| \,\Big| = s \tag{7.5.1}$$

where the outer absolute value essentially allows for re-ordering of the points. Kaya, Akça, Günalti, and Özcan document almost 20 different cases of hyperbolas with most cases producing planar regions or degenerate hyperbolas. Even sorting out the non-degenerate cases is daunting. One case is outlined below.

$|x_2 - x_1 - (y_2 - y_1)| < s < d_f$

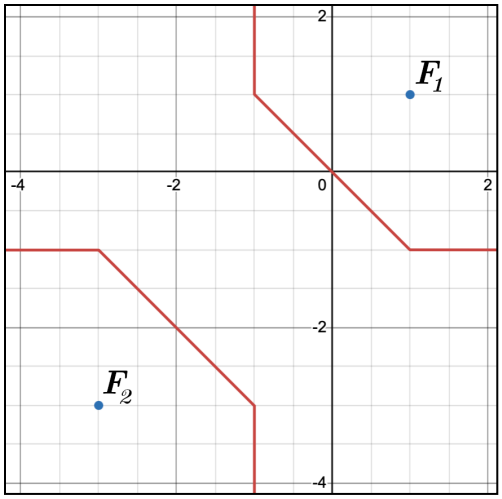


Figure 7.5.1 A non-degenerate two-foci taxicab hyperbola

$$\Big|\, |x - 1| + |y - 1| - |x + 3| - |y + 3| \,\Big| = 4$$

The Focus-Directrix Definition

Let $F = (h, k)$ be the focus and l the directrix for a focus-directrix hyperbola. Then for each point $P = (x, y)$ on the curve, the ratio of the focus distance to the directrix distance (the eccentricity) is a constant e greater than one. Using Theorem 2.3.1 for the distance from a point to a line of the form $ax + by + c = 0$, we have

$$\frac{d(P,F)}{d(P,l)} = e > 1$$

$$|x-h| + |y-k| = e \cdot \frac{|ax+by+c|}{\max\{|a|, |b|\}} \tag{7.5.2}$$

This is the same formulation as for a focus-directrix parabola or ellipse but now the eccentricity is greater than unity. As with ellipses, the slant of the directrix plays an important role in characteristics of the resulting hyperbola.

Case 1: Vertical directrix. One line of symmetry.

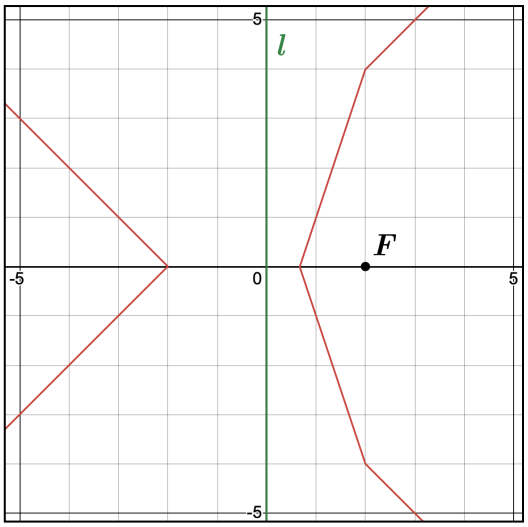


Figure 7.5.2 A focus-directrix taxicab hyperbola (red) with a vertical directrix ($F = (2, 0)$, $e = 2$, **directrix**: $x = 0$) defined by:

$$|x-2| + |y| = 2 \cdot |x|$$

Case 2: Diagonal directrix. One axis of symmetry.

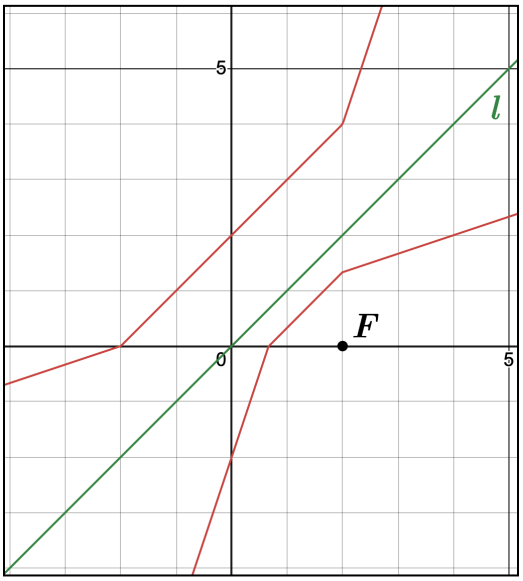


Figure 7.5.3 A focus-directrix taxicab hyperbola (red) with a diagonal directrix ($F = (2, 0)$, $e = 2$, **directrix**: $y = x$) defined by:

$$|x-2| + |y| = 2 \cdot |x-y|$$

Case 3: Arbitrary directrix. These hyperbolas do not have any symmetry and break into three sub-cases depending on the relative values of the eccentricity and absolute value of the slope of the directrix.

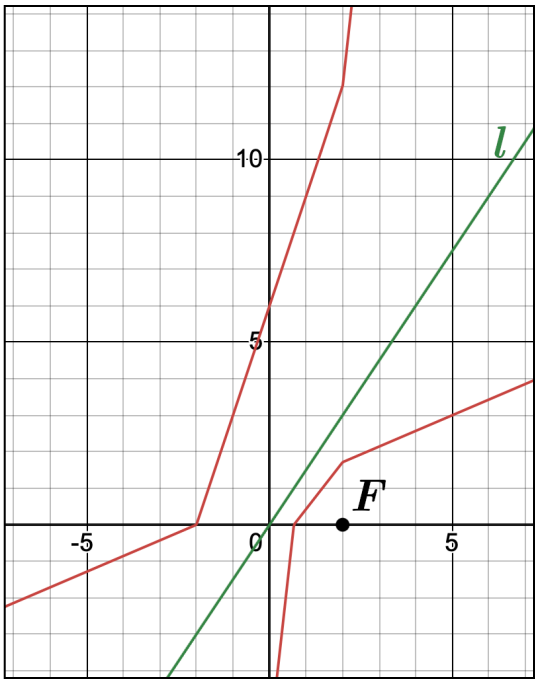


Figure 7.5.4 A focus-directrix taxicab hyperbola (red) with absolute directrix slope less than eccentricity ($F = (2,0)$, $e = 2$, **directrix**: $y = 3x/2$) defined by:

$$|x-2| + |y| = 2 \cdot \frac{|3x-2y|}{3}$$

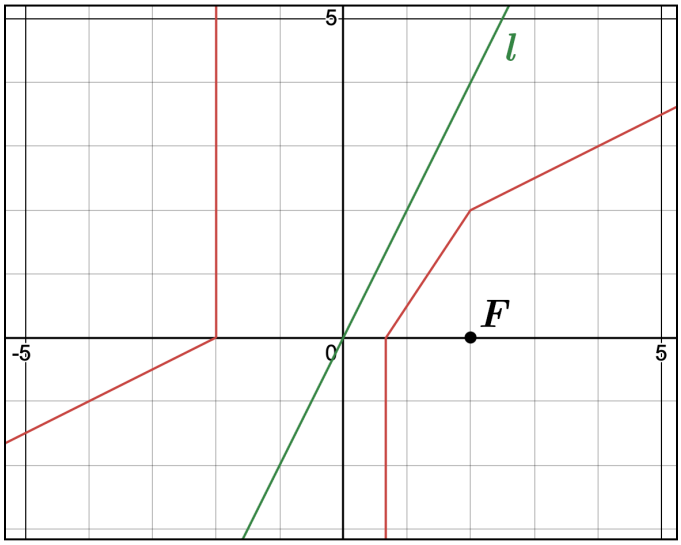


Figure 7.5.5 A focus-directrix taxicab hyperbola (red) with absolute directrix slope equal to eccentricity ($F = (2,0)$, $e = 2$, **directrix**: $y = 2x$) defined by:

$$|x-2| + |y| = 2 \cdot \frac{|2x-y|}{2}$$

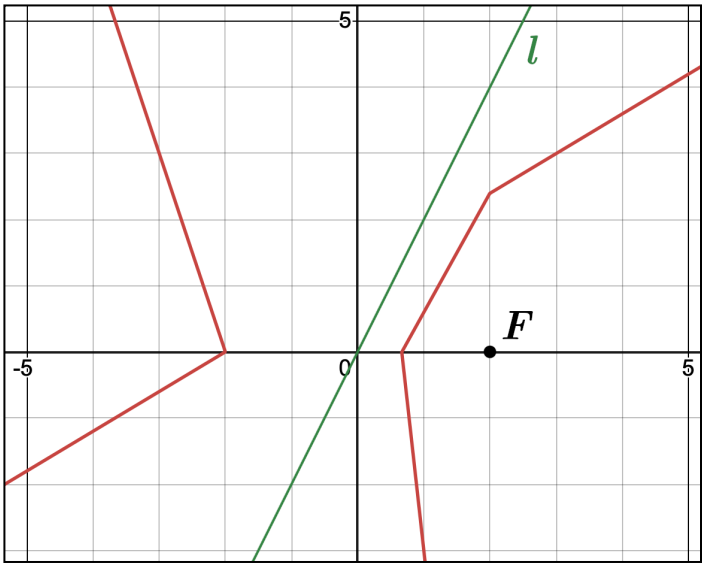


Figure 7.5.6 A focus-directrix taxicab hyperbola (red) with absolute directrix slope greater than eccentricity ($F = (2,0)$, $e = 2$, **directrix**: $y = 3x$) defined by:

$$|x-2| + |y| = 2 \cdot \frac{|3x-y|}{3}$$

Hyperbolas as Conic Sections

Of particular note here is a distinction with Euclidean geometry. Euclidean hyperbolas are formed from intersecting the surface of a right circular cone and a *vertical* plane. If the plane is not vertical a parabola is produced. The various Euclidean hyperbolas are produced by varying the distance the plane is from the axis of the cones. In taxicab geometry, the intersecting plane is usually not vertical; it only needs to be steep enough intersect both halves of the double cone. In part, it is this non-vertical nature of the intersecting plane that causes taxicab hyperbolas to not be symmetric and indeed have very different shapes for the two halves of the curve, also in contrast to Euclidean hyperbolas which have a great deal of symmetry.

A further reminder here as well that the curves formed on the surface of the taxicab right circular cone must be projected down to the xy-plane in order to be equivalent to taxicab focus-directrix hyperbolas.

Section 7.6: General Formula for Taxicab Conics

As seen in the previous sections, the equations for the various conics have many commonalities. So, as in Euclidean geometry, it is possible to formulate a general equation describing a taxicab conic. This formulation was developed by Kaya, Akça, Günalti, and Özcan in 2000.

The general equation of a taxicab conic with either 1) two foci $F_1 = (x_1, y_1)$ and $F_2 = (x_2, y_2)$, or 2) a focus $F = (x_1, y_1)$ and directrix $ax + by + c = 0$ is given by

$$|x - x_1| + |y - y_1| + \alpha(|x - x_2| + |y - y_2|) + \beta|ax + by + c| = \pm\gamma \qquad (7.6.1)$$

where

- $\alpha \in \{-1,0,1\}$ is a dual selector between the two-focus ($\alpha = \pm 1$) and focus-directrix ($\alpha = 0$) cases, and the focus distance sum ($\alpha = 1$ for a parabola) and difference ($\alpha = -1$ for a hyperbola) cases
- β is a combined selector between the two-focus ($\alpha = \pm 1$ giving $\beta = 0$) and focus-directrix ($\alpha = 0$) cases and a discriminator between the shallow and steep slope directrix cases:
  $\beta = \dfrac{e(\alpha^2 - 1)}{\max\{|a|, |b|\}}$
  where e is the eccentricity of the conic
- $\gamma \geq 0$ is a constant

While this formulation is interesting, quite complete, and an accomplishment, I have a relatively strong preference against it for two reasons. First and foremost, it obscures the strong distinction between the two-foci and focus-directrix approaches in taxicab geometry. As mentioned elsewhere in this chapter, in Euclidean geometry all roads to conics (conical sections, two-foci, and focus-directrix definitions) arrive at the same destination. In taxicab geometry, the destinations are very different. Two-foci conics are completely different curves than focus-directrix conics and conical sections in taxicab geometry. By combining all of these into one equation, I feel that this distinction is relegated to the background and could leave the impression that such an equation is the taxicab equivalent to the Euclidean general conic equation when in many ways it is not.

Second, this equation describes more degenerate and planar-region "conics" than it does "normal" conics. Most of these are centered on the two-foci cases which, beyond their disconnect from conical sections, is an additional reason to relegate this formulation of taxicab "conics" to the background. This reason will obviously come down to quite a bit of personal preference and direction of interest, so it is a less compelling reason for me to lean away from this formulation than the first reason given above.

Section 7.7: The Directrix Incidence Property

For "true" taxicab conic sections, those arising from the focus-directrix definitions, a curious incidence property arises, one that incredibly involves FOUR lines passing through key parts of the conic section. This was first noted by Robert Laatsch.

Theorem 7.7.1 *For a taxicab conic arising from a focus-directrix definition, the following lines will intersect the directrix at a common point: lines passing through two alternate branches of the conic and a line of slope* ± 1 *through the focus.*

We will take the conics in turn to prove this incidence property which will also clarify more precisely what is meant by "alternate branches."

Parabolas

Case 1: A vertical or horizontal directrix. For simplicity, assume a parabola with horizontal directrix $y = 0$ and focus $F = (h, k)$ shown in Figure 7.7.1. Then the equation of the parabola is

$$|x - h| + |y - k| = |y|$$

Resolving the absolute value in cases, we can derive the actual equations of the four lines composing the parabola.

1) $x > h$, $y > k$: The equation simplifies to $x = h + k$. The extension of this branch intersects the directrix at $(h + k, 0)$.
2) $x > h$, $y < k$: The equation simplifies to $y = \frac{1}{2}x + \frac{k-h}{2}$. Setting $y = 0$ we see that the extension of this branch intersects the directrix at $(h - k, 0)$.
3) $x < h$, $y < k$: The equation simplifies to $y = -\frac{1}{2}x + \frac{k+h}{2}$. Setting $y = 0$ we see that the extension of this branch intersects the directrix at $(h + k, 0)$.
4) $x < h$, $y > k$: The equation simplifies to $x = h - k$. The extension of this branch intersects the directrix at $(h - k, 0)$.

Lines of slope ± 1 through the focus have the form:

5) $y = x - h + k$ which intersects the directrix at $(h - k, 0)$.
6) $y = -x + h + k$ which intersects the directrix at $(h + k, 0)$.

Using items 1, 3, and 6, we see that three lines intersect the directrix at $Q = (h + k, 0)$ and with items 2, 4, and 5 three lines intersect the directrix at $Q' = (h - k, 0)$, in each case taking "alternating" branches of the conic.

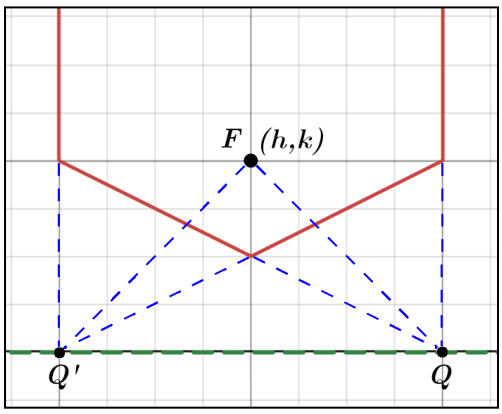


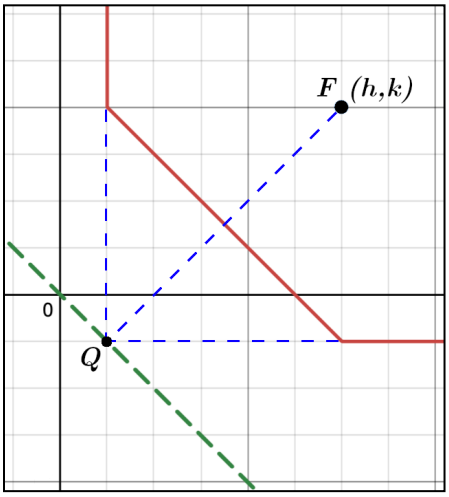
Figure 7.7.1 Directrix incidence property for a (focus-directrix) taxicab parabola with horizontal directrix (green)

Case 2: A diagonal directrix. For simplicity, select $y = -x$ for the directrix with focus $F = (h, k)$ above the directrix:

$$|x - h| + |y - k| = |x + y|$$

Since the focus is above the directrix, $x + y > 0$. Again, taking the absolute value cases in turn:

1) $x > h$, $y < k$: The equation is $y = \frac{k-h}{2}$ and intersects the directrix at $\left(\frac{h-k}{2}, \frac{k-h}{2}\right)$.
2) $x < h$, $y < k$: The equation is $y = -x + \frac{k+h}{2}$ which is parallel to the directrix and therefore does not play a part in the directrix incidence property.
3) $x < h$, $y > k$: The equation is $x = \frac{h-k}{2}$ and intersects the directrix at $\left(\frac{h-k}{2}, \frac{k-h}{2}\right)$.



Only a line of slope 1 through the focus intersects the directrix. The equation of the line is $y = x + k - h$ which intersects the directrix at $\left(\frac{h-k}{2}, \frac{k-h}{2}\right)$. In this case, there is only one incidence point $Q = \left(\frac{h-k}{2}, \frac{k-h}{2}\right)$ on the directrix.

Figure 7.7.2 Directrix incidence property for a (focus-directrix) taxicab parabola with diagonal directrix (green)

Case 3: An arbitrary sloped directrix. For simplicity, assume a steep directrix through the origin ($ax + by = 0$) with the focus above the line (Figure 7.7.3). Then $a > b$ and for all points on the parabola $ax + by > 0$.

1) $x > h$, $y < k$: The equation is $y = \frac{a(k-h)}{a+b}$ and intersects the directrix at $\left(\frac{b(h-k)}{a+b}, \frac{a(k-h)}{a+b}\right)$.
2) $x < h$, $y < k$: The equation is $y = -\frac{2a}{a+b}x + \frac{a(h+k)}{a+b}$ but it is not one of the alternating branches of the conic.
3) $x < h$, $y > k$: The equation is $y = \frac{2a}{a-b}x + \frac{a(k-h)}{a-b}$ and intersects the directrix at $\left(\frac{b(h-k)}{a+b}, \frac{a(k-h)}{a+b}\right)$.

Only a line of slope 1 through the focus intersects the directrix. The equation of the line is $y = x + k - h$ which intersects the directrix at $\left(\frac{b(h-k)}{a+b}, \frac{a(k-h)}{a+b}\right)$.

In this case, there is only one incidence point $Q = \left(\frac{b(h-k)}{a+b}, \frac{a(k-h)}{a+b}\right)$ on the directrix.

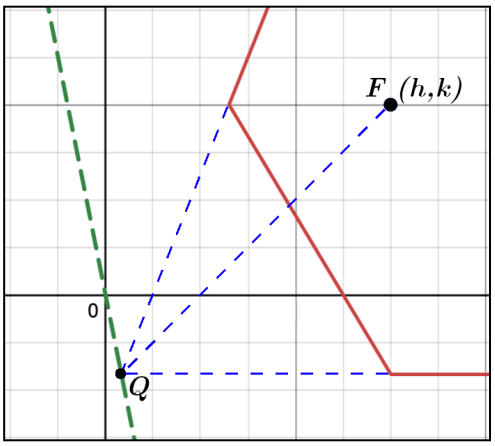


Figure 7.7.3 Directrix incidence property for a (focus-directrix) taxicab parabola with arbitrary sloped directrix (green)

Ellipses

Case 1: A vertical or horizontal directrix. For simplicity, assume an ellipse with vertical directrix $x = c$, focus $F = (h, k)$ to the right of the directrix, and eccentricity $e < 1$ shown in Figure 7.7.4. Then the equation of the ellipse is

$$|x - h| + |y - k| = e|x - c|$$

A taxicab ellipse is composed of four line segments.

1) $x > h$, $y > k$: Equation: $y = (e - 1)x + h + k - ec$, directrix intersection: $(c, h + k - c)$.
2) $x > h$, $y > k$: Equation: $y = (1 - e)x + k - h + ec$, directrix intersection: $(c, k - h + c)$.
3) $x > h$, $y > k$: Equation: $y = -(e + 1)x + h + k + ec$, directrix intersection: $(c, h + k - c)$.
4) $x > h$, $y > k$: Equation: $y = (e + 1)x + k - h - ec$, directrix intersection: $(c, k - h + c)$.

Lines of slope ± 1 through the focus have the form:

5) $y = x + k - h$ which intersects the directrix at $(c, k - h + c)$.
6) $y = -x + h + k$ which intersects the directrix at $(c, h + k - c)$.

Pairing items 1-3-6 gives a point of incidence on the directrix at $Q = (c, h + k - c)$ and pairing items 2-4-5 yields a point of incidence on the directrix at $Q' = (c, k - h + c)$.

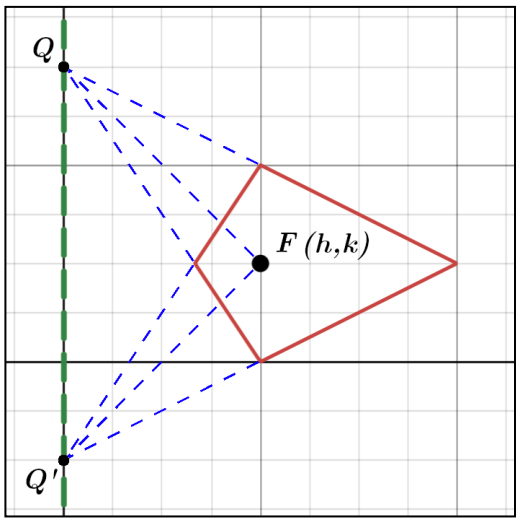


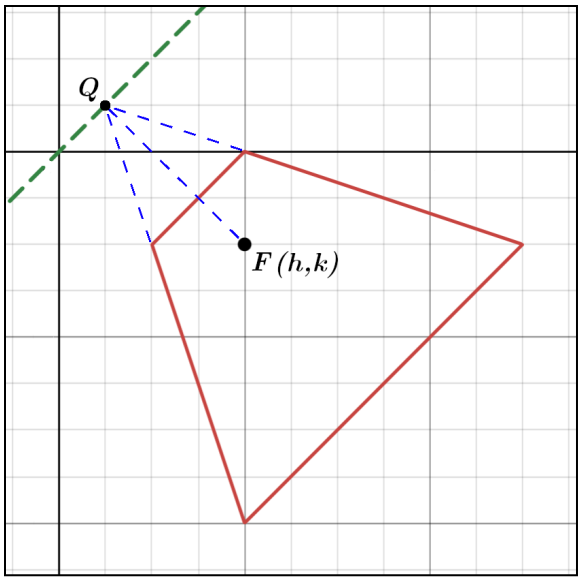
Figure 7.7.4 Directrix incidence property for a (focus-directrix) taxicab ellipse with vertical directrix (green)

Case 2: A diagonal directrix. For simplicity, select $y = x$ for the directrix with focus $F = (h, k)$ below the directrix (Figure 7.7.5):

$$|x - h| + |y - k| = e|x - y + c|$$

Since the focus is below the directrix, $x - y + c > 0$. Again, taking the cases in turn:

1) $x > h$, $y > k$: Equation: $y = \frac{e-1}{e+1}x + \frac{h+k+ec}{e+1}$, directrix intersection: $\left(\frac{h+k-c}{2}, \frac{h+k+c}{2}\right)$.
2) $x > h$, $y > k$: Equation: $y = x + \frac{h-k+ec}{e-1}$ which is parallel to the directrix and therefore does not play a part in the directrix incidence property
3) $x > h$, $y > k$: Equation: $y = \frac{e+1}{e-1}x + \frac{ec-h-k}{e-1}$, directrix intersection: $\left(\frac{h+k-c}{2}, \frac{h+k+c}{2}\right)$.
4) $x > h$, $y > k$: Equation: $y = x + \frac{k-h+ec}{e+1}$ which is parallel to the directrix and therefore does not play a part in the directrix incidence property.



Only a line of slope -1 through the focus intersects the directrix. The equation of the line is $y = -x + k + h$ which intersects the directrix at $\left(\frac{h+k-c}{2}, \frac{h+k+c}{2}\right)$.

In this case, there is only one incidence point $Q = \left(\frac{h+k-c}{2}, \frac{h+k+c}{2}\right)$ on the directrix.

Figure 7.7.5 Directrix incidence property for a (focus-directrix) taxicab ellipse with diagonal directrix (green)

Case 3: An arbitrary sloped directrix. For simplicity, assume a shallow directrix through the origin ($ax + by = 0$) with the focus below the line (Figure 7.7.6). Then $a < b$ and for all points on the ellipse $ax + by < 0$:

$$|x - h| + |y - k| = e\frac{|ax + by|}{b}$$

1) $x > h$, $y > k$: Equation: $y = \frac{ae - b}{b(e+1)}x + \frac{h+k}{e+1}$, directrix intersection: $(\frac{b(h+k)}{b-a}, \frac{a(h+k)}{a-b})$.
2) $x > h$, $y < k$: Equation: $y = -\frac{ae + b}{b(e-1)}x + \frac{h-k}{e-1}$, directrix intersection: $(\frac{b(h-k)}{a+b}, \frac{a(k-h)}{a+b})$.
3) $x < h$, $y < k$: Equation: $y = \frac{b - ae}{b(e-1)}x - \frac{h+k}{e-1}$, directrix intersection: $(\frac{b(h+k)}{b-a}, \frac{a(h+k)}{a-b})$.
4) $x < h$, $y > k$: Equation: $y = \frac{b - ae}{b(e+1)}x + \frac{k-h}{e+1}$, directrix intersection: $(\frac{b(h-k)}{a+b}, \frac{a(k-h)}{a+b})$.

Lines of slope ± 1 through the focus have the form:

5) $y = x + k - h$ which intersects the directrix at $(\frac{b(h-k)}{a+b}, \frac{a(k-h)}{a+b})$.
6) $y = -x + h + k$ which intersects the directrix at $(\frac{b(h+k)}{b-a}, \frac{a(h+k)}{a-b})$.

Pairing items 1-3-6 gives a point of incidence on the directrix at $Q = (\frac{b(h+k)}{b-a}, \frac{a(h+k)}{a-b})$ and pairing items 2-4-5 yields a point of incidence on the directrix at $Q' = (\frac{b(h-k)}{a+b}, \frac{a(k-h)}{a+b})$.

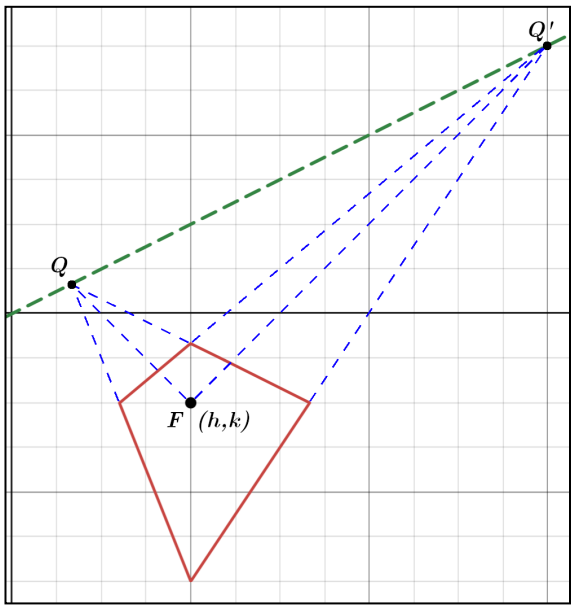


Figure 7.7.6 Directrix incidence property for a (focus-directrix) taxicab ellipse with arbitrary slanted directrix (green)

It should be noted that in all of the ellipse cases the point(s) of incidence on the directrix are not dependent on the eccentricity of the ellipse, only the relative positions and orientations of the focus and directrix.

Hyperbolas

Case 1: A vertical or horizontal directrix. For simplicity, assume a hyperbola with vertical directrix $x = c$, focus $F = (h, k)$ to the right of the directrix, and eccentricity $e > 1$ shown in Figure 7.7.7. Then the equation of the ellipse is

$$|x - h| + |y - k| = e \cdot |x - c|$$

A taxicab hyperbola of this type is composed of six line segments.

1) $x > h$, $y > k$: Equation: $y = (e - 1)x + h + k - ec$, directrix intersection: $(c, h + k - c)$.
2) $x > h$, $y > k$: Equation: $y = (1 - e)x + k - h + ec$, directrix intersection: $(c, k - h + c)$.
3) $c < x < h$, $y > k$: Equation: $y = (e + 1)x + k - h - ec$, dir. intersection: $(c, k - h + c)$.
4) $c < x < h$, $y < k$: Equation: $y = -(e + 1)x + h + k + ec$, dir. intersection: $(c, h + k - c)$.
5) $x < c < h$, $y > k$: Equation: $y = (1 - e)x + k - h + ec$, dir. intersection: $(c, k - h + c)$.
6) $x < c < h$, $y < k$: Equation: $y = (e - 1)x + h + k - ec$, dir. intersection: $(c, h + k - c)$.

Lines of slope ± 1 through the focus have the form:

7) $y = x + k - h$ which intersects the directrix at $(c, k - h + c)$.
8) $y = -x + h + k$ which intersects the directrix at $(c, h + k - c)$.

Pairing items 1-4-6-8 gives a point of incidence on the directrix at $Q = (c, h + k - c)$ and pairing items 2-3-5-7 yields a point of incidence on the directrix at $Q' = (c, k - h + c)$. Notice the taxicab circle of radius $h - c$ formed by the focus, focus image in the directrix, and the directrix incidence points.

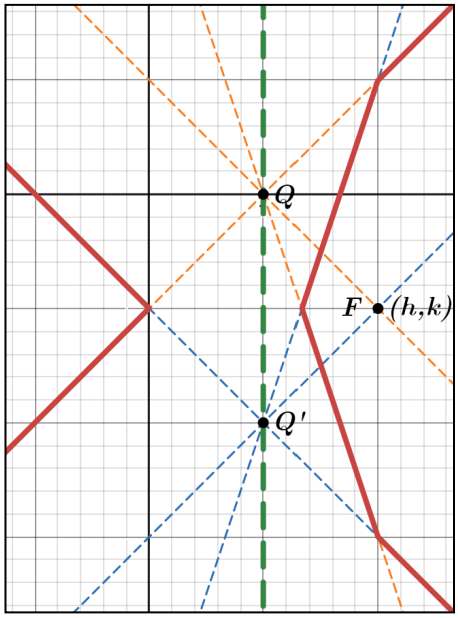


Figure 7.7.4 Directrix incidence property for a (focus-directrix) taxicab hyperbola (red) with vertical directrix (green)

Case 2: A diagonal directrix. For simplicity, select $x - y + c = 0$ for the directrix with focus $F = (h, k)$ below the directrix (Figure 7.7.5):

$$|x - h| + |y - k| = e\,|x - y + c|$$

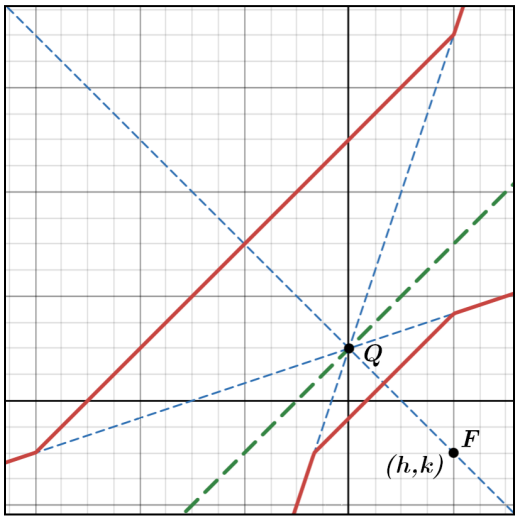
Since the focus is below the directrix, $x - y + c > 0$. Because of the symmetry present, two equations describe the relevant four branches of the hyperbola with the remaining two branches being parallel to the directrix and therefore not relevant to directrix incidence.

1) $x > h,\ y > k,\ y < x + c$ OR $x < h,\ y < k,\ y > x + c$:
Equation: $y = \frac{e-1}{e+1}x + \frac{h+k+ec}{e+1}$, directrix intersection: $\left(\frac{h+k-c}{2}, \frac{h+k+c}{2}\right)$ from 2 branches

2) $x < h,\ y < k,\ y < x + c$ OR $x > h,\ y > k,\ y > x + c$:
Equation: $y = \frac{e+1}{e-1}x + \frac{h+k-ec}{1-e}$, directrix intersection: $\left(\frac{h+k-c}{2}, \frac{h+k+c}{2}\right)$ from 2 branches

3) $x < h,\ y > k,\ y < x + c$:
Equation: $y = x + \frac{k-h+ec}{e+1}$ which is parallel to the directrix and therefore does not play a part in the directrix incidence property

4) $x < h,\ y > k,\ y > x + c$:
Equation: $y = x + \frac{h-k+ec}{e-1}$ which is parallel to the directrix and therefore does not play a part in the directrix incidence property



Only a line of slope -1 through the focus intersects the directrix. The equation of the line is $y = -x + k + h$ which intersects the directrix at $\left(\frac{h+k-c}{2}, \frac{h+k+c}{2}\right)$.

In this case, there is only one incidence point $Q = \left(\frac{h+k-c}{2}, \frac{h+k+c}{2}\right)$ on the directrix.

Figure 7.7.5 Directrix incidence property for a (focus-directrix) taxicab hyperbola (red) with diagonal directrix (green)

Case 3:

As with ellipses, for hyperbolas the point(s) of incidence on the directrix are not dependent on the eccentricity of the hyperbola, only the relative positions and orientations of the focus and directrix.

Observing the Directrix Incidence Property

Laatsch took an alternate approach to observing this property, although for brevity he did not outline an overt proof. He noted that these incidence points on the directrix are a natural consequence of the projection of the plane intersecting the taxicab right circular cone (pyramid) onto a horizontal plane (parallel to the xy-plane) and such a four-line incidence property is therefore not surprising in this geometry. The reader is encouraged to examine his thought process in his research paper which is cited in the References.

Section 7.8: Apollonian Sets

Although not strictly a part of conic sections, Apollonian sets are a related concept. Given two points a and b in the plane and a positive number $k \neq 1$, the Apollonian set is defined as the points P that satisfy

$$\frac{d(A,P)}{d(B,P)} = k$$

This would be most similar to the focus-directrix definition of a parabola with eccentricity k except the directrix has been replaced with a second focus.

In Euclidean geometry an Apollonian set is always a circle. In taxicab geometry, not so much, as was first discovered by Eric Bahuaud, et al. in 2020.

If the points A and B lie along a diagonal, then the Apollonian set is always an isosceles trapezoid with specific properties. An example is shown in Figure 7.8.1 for $A = (0,0)$, $B = (1,1)$, and $k = \frac{3}{2}$.

Theorem 7.8.1 *If points A and B lie along a diagonal with $k > 1$, then the Apollonian set is an isosceles trapezoid with the following properties:*

- *the line of symmetry is the diagonal through A and B*
- *the vertices lie along the horizontal and vertical line through point B*
- *extensions of the legs of the trapezoid lines intersect at point A*
- *the slopes of the legs are $m\frac{k+1}{k-1}$ and $m\frac{k-1}{k+1}$, where m is the slope of the diagonal through A and B*

If the points A and B do not lie along a diagonal, then the Apollonian set is always a kind of "union" of two "simple" Apollonian sets generated using Theorem 7.8.1 with point B and two other points diagonal from B. There are some complications to be overcome in terms of expressing this "union" along with defining the two related points.

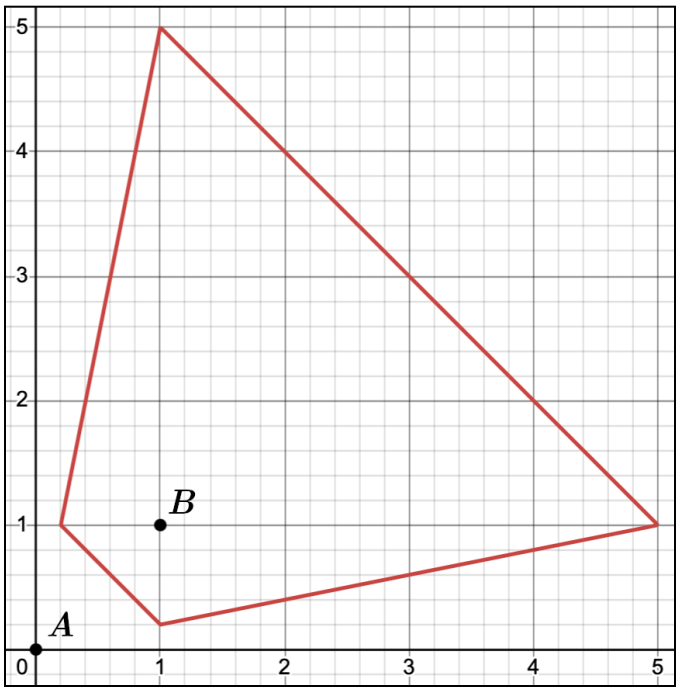


Figure 7.8.1 An Apollonian set (red) for two points on a diagonal

An Apollonian set is always a closed figure. Consider shading the interior of this closed figure. We will call this a filled Apollonian set. When defining the final Apollonian set for points that do not lie on a diagonal, we will find two filled Apollonian sets, union these regions, and then take the boundary of the filled union. This is the way in which two overlapping Apollonian sets will be combined.

Let A and B be two points that do not lie on a diagonal. In particular, we will walk through this process for points $A = (0, 0)$ and $B = (6, 3)$ with $k = \frac{3}{2}$ as shown in Figure 7.8.2. Construct lines of slope ± 1 through both points (red dashed lines). This will create a rectangle with A and B vertices at opposite ends of one of the diagonals. Define the other vertices to be A' (blue star) and B' (green star). These two points lie along diagonals from point B. Using Theorem 7.8.1, construct the Apollonian sets for the pairs $A' - B$ (blue dashed line) and $B' - B$ (green dotted line) with the same value for k. Fill the two Apollonian sets, union the filled sets, and take the boundary (orange line). This is the Apollonian set for A and B.

In many cases one of the trapezoids will be fully contained in the other, and so the Apollonian set looks the same as if the points were on a diagonal. But, in other cases such as this one, the smaller trapezoid is not fully contain in the larger and so there is a "kink" added the larger trapezoid. But, in all cases, the Apollonian set will always be a union of the two "simple" Apollonian sets generated from points along diagonals from B.

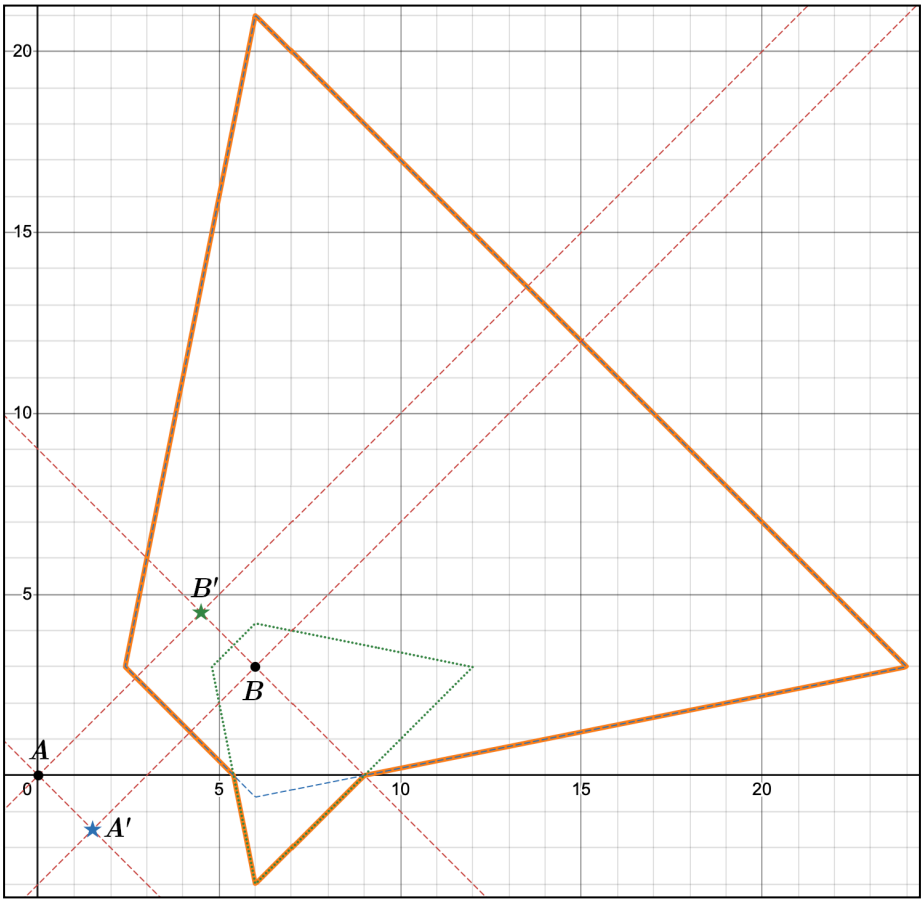


Figure 7.8.2 Construction of an Apollonian set (orange) for two points not on a diagonal

An interesting case occurs when the points lie along a horizontal or vertical line. In this case, the union of the Apollonian sets (dashed blue and green lines) becomes a kite (see Figure 7.8.3 with $A = (0, 0)$, $B = (2, 0)$, and $k = \frac{3}{2}$).

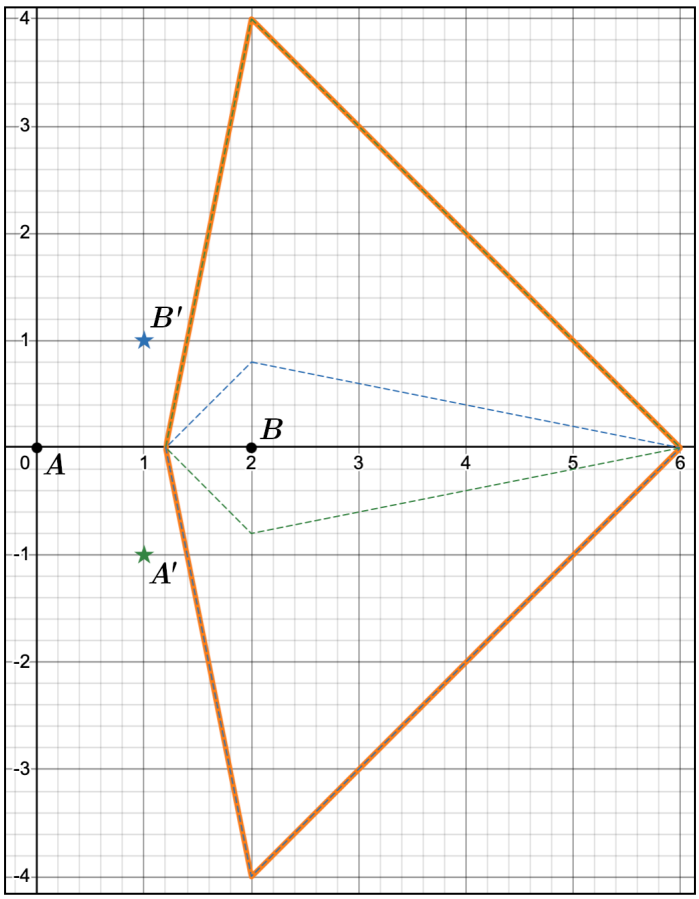


Figure 7.8.3 Construction of an Apollonian set (orange) for two points on a horizontal line

Chapter 8
Calculus

Section 8.1: Arc Length

As discussed in Section 3.1, taxicab length when measured in two (or higher) dimensions will differ from Euclidean length. Therefore, if we consider a random curve in two or three dimensions, the traditional Euclidean formula for arc length will need to be adjusted to get a taxicab expression of the length.

In Euclidean geometry, the arc length of a two-dimensional curve described by a function f with a continuous derivative over an interval $[a, b]$ is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

To find a corresponding formula in taxicab geometry, we may follow a traditional method for deriving arc length. First, split the interval $[a, b]$ into n subintervals with widths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ by defining points $x_1, x_2, \ldots, x_{n-1}$ between a and b (Figure 8.1.1).

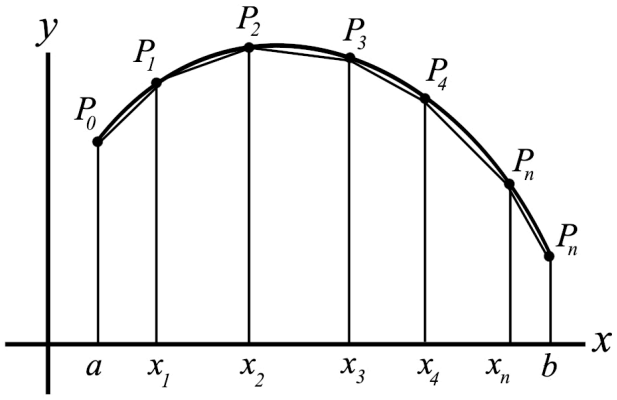


Figure 8.1.1 Approximating arc length in taxicab geometry.

This allows for definition of points $P_0, P_1, P_2, \ldots, P_n$ on the curve whose x-coordinates are $a, x_1, x_2, \ldots, x_{n-1}, b$. By connecting these points, we obtain a polygonal path approximating the curve. At this crucial point of the derivation we apply the taxicab metric instead of the Euclidean metric to obtain the taxicab length L_k of the kth line segment,

$$L_k = \Delta x_k + |f(x_k) - f(x_{k-1})|$$

By the Mean Value Theorem, there is a point x_k^* between x_{k-1} and x_k such that $f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$. Thus, the taxicab length of the kth segment becomes

$$L_k = \Delta x_k + |f'(x_k^*)\Delta x_k| = (1 + |f'(x_k^*)|)\,\Delta x_k$$

and the taxicab length of the entire polygonal path is

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} (1 + |f'(x_k^*)|)\,\Delta x_k$$

By increasing the number of subintervals while forcing $\max \Delta x_k \to 0$, the length of the polygonal path will approach the arc length of the curve. So,

$$L = \lim_{\max \Delta x \to 0} \sum_{k=1}^{n} (1 + |f'(x_k^*)|)\, \Delta x_k$$

The right side of this equation is a definite integral, so the taxicab arc length of the curve described by the function f over the interval $[a, b]$ is given by

$$L = \int_a^b \left(1 + |f'(x)|\right)\, dx \tag{8.1.1}$$

As an example, the northeast quadrant of a taxicab circle of radius r centered at the origin is described by $f(x) = -x + r$. The arc length of this curve over the interval $[0, r]$ is given by

$$L = \int_0^r \left(1 + \left|-1\right|\right)\, dx = 2r$$

which is precisely one-fourth of the circumference of a taxicab circle of radius r. A more interesting application involves the northeast quadrant of the Euclidean circle of radius r at the origin described by $f(x) = \sqrt{r^2 - x^2}$.

$$\begin{aligned} L &= \int_0^r \left(1 + \left|-x\left(r^2 - x^2\right)^{-\frac{1}{2}}\right|\right) dx \\ &= \left(x - \sqrt{r^2 - x^2}\right)\Big|_0^r \\ &= r - 0 - 0 + r \\ &= 2r \end{aligned}$$

The distance is the same as if we had traveled along the taxicab circle between the two endpoints! While this is a shocking result to the Euclidean observer who is accustomed to

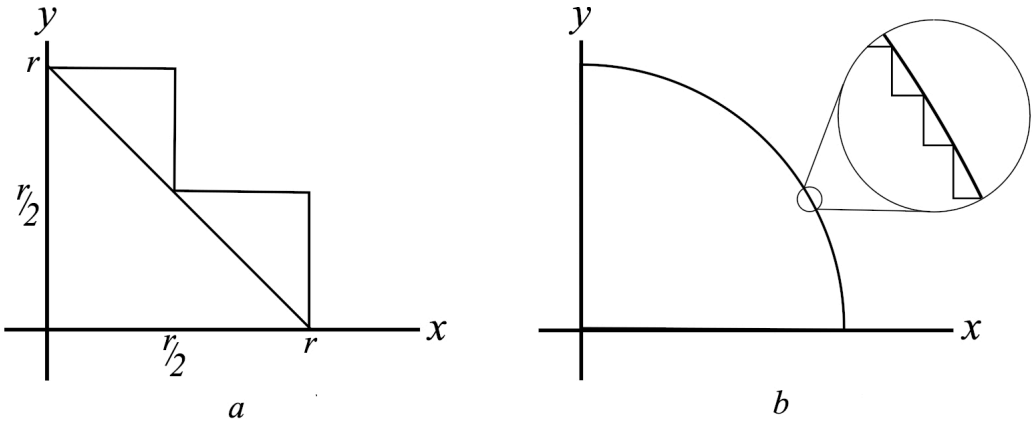


Figure 8.1.2 a) Multiple straight-line paths between two points can have the same length in taxicab geometry. b) A curve can be approximated by small horizontal and vertical paths.}

distinct paths between two points generally having different lengths, the taxicab observer merely shrugs his shoulders. A curve such as a Euclidean circle can be approximated with increasingly small horizontal and vertical steps near its path (Figure 8.1.2b). As illustrated in Section 2.1, multiple straight-line paths between two points have the same length in taxicab geometry (see Figure 8.1.2a). So, each of these approximations of the Euclidean circle will have the same taxicab length. Therefore, we should expect the limiting case to also have the same length. To further make the point, we can follow the Euclidean parabola

$$f(x) = -\frac{1}{r}x^2 + r$$

in the first quadrant and arrive at the same result (Figure 8.1.3).

$$L = \int_0^r \left(1 + \left|\frac{-2x}{r}\right|\right) dx = r + r = 2r$$

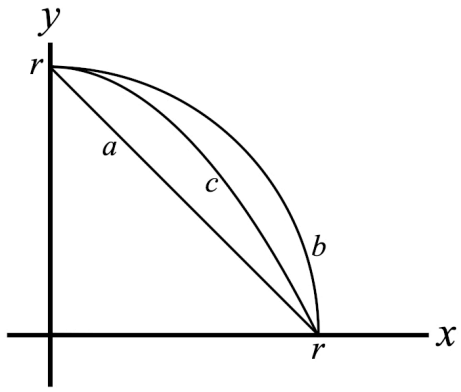


Figure 8.1.3 Multiple curves between two points can have the same length in taxicab geometry. Shown are a) part of the taxicab circle of radius r, b) part of the Euclidean circle of radius r, and c) part of the curve $-\frac{1}{r}x^2 + r$

To formalize these observations, we have the following theorem.

Theorem 8.1.1 *If a function f is monotone increasing or decreasing and differentiable with a continuous derivative over an interval $[a,b]$, then the taxicab arc length of f over $[a,b]$ is*

$$L = (b-a) + \left|f(b) - f(a)\right| \tag{8.1.2}$$

(i.e. the path from $(a, v = f(a))$ to $(b, w = f(b))$ is independent of the function f under the stated conditions).

PROOF: Using Equation 8.1.1, the taxicab arc length of f over $[a,b]$ is

$$L = \int_a^b \left(1 + \left|f'(x)\right|\right) dx = (b-a) + \int_a^b \left(\left|f'(x)\right|\right) dx$$

If f is monotone increasing, $f'(x) \geq 0$. And, since f has a continuous derivative, we may apply the First Fundamental Theorem of Calculus to get

$$L = \left(b - a\right) + f\left(b\right) - f\left(a\right)$$

If f is monotone decreasing, we get the similar result

$$L = \left(b - a\right) + f\left(a\right) - f\left(b\right)$$

In either case, the latter difference must be positive so we have

$$L = \left(b - a\right) + \left|f\left(b\right) - f\left(a\right)\right|$$

□

For functions that are not monotone increasing or decreasing, the arc length can be found using this theorem by taking the sum of the arc lengths over the subintervals where the function is monotone increasing or decreasing.

The simplest extension of the arc length of a curve to three dimensions is for parametric curves. For a curve in two dimensions, we can parameterize the function $f = (x(t), y(t))$ and modify Equation 8.1.1 to get

$$L = \int_a^b \left(\left| \frac{dx}{dt} \right| + \left| \frac{dy}{dt} \right| \right) dt$$

Generalizing this to three dimensions, we have

$$L = \int_a^b \left(\left| \frac{dx}{dt} \right| + \left| \frac{dy}{dt} \right| + \left| \frac{dz}{dt} \right| \right) dt$$

Intuitively, Theorem 8.1.1 generalizes to three-dimensions saying that the length of a three-dimensional "monotonic" curve depends only on its endpoints.

Section 8.2: Area Under a Curve

The standard approach to the computation of two-dimensional area in under-graduate calculus courses is integration. Estimates of area under a functional curve involve increasing numbers of rectangles that each approximate a portion of the area under the curve. Since Euclidean and taxicab geometry agree on length in the horizontal and vertical directions (Section 3.2), these area approximations in the two geometries will agree. Therefore, we should expect the calculation of area under a curve by integration to transfer seamlessly to taxicab geometry. This is a further argument to view area in two dimensions consistently between the two geometries.

Section 8.3: Surfaces of Revolution

In her unpublished dissertation "Taxicab Geometry: Not the Shortest Ride Around Town" at Iowa State University in 2007, Christina Janssen makes several attempts to revolve half a taxicab circle about an axis to obtain a taxicab sphere. The first approach revolves the upper half of a taxicab circle in a *Euclidean* manner about the horizontal axis resulting in a "sphere" with some straight edges but also with some curved surfaces (Figure 8.3.1a). This does not seem to quite fit the "squarish" nature of seemingly everything in taxicab geometry and does not appear to satisfy the usual definition of a sphere (all points equidistant, in a taxicab sense, from the center).

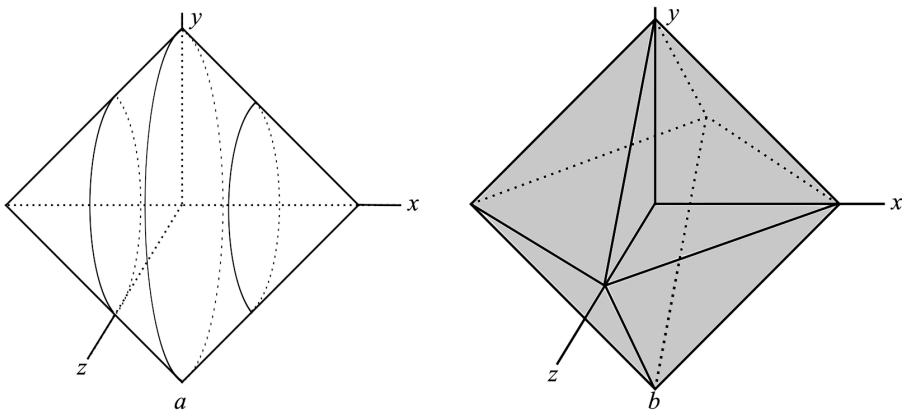


Figure 8.3.1 A Janssen taxicab "sphere" and b) a taxicab sphere.

However, the author's creativity is to be highly commended and her line of thought is quite instructive. If we consider a Euclidean solid of revolution about the x-axis, the cross-section of the solid in the yz-plane is a Euclidean circle. Therefore, for a taxicab revolution about the x-axis, the cross-section should be a taxicab circle in that plane. A taxicab sphere obtained by revolving a taxicab circle about the horizontal axis in a taxicab manner would yield Figure 8.3.1b. This completely "squarish" object seems to fit the geometry better and indeed does, by construction, satisfy the usual definition of a sphere. So, revolution about the horizontal axis should follow the path of a *taxicab* circle in the yz-plane and not the path of a Euclidean circle.

As an application of our approach to surface area in three dimensions, we would like to find a formula for the surface area of a solid of revolution. For a differentiable function f over an interval $[a, b]$ divide the interval into n subintervals. Using a linear approximation of the curve over each subinterval, we revolve each line segment about the x-axis. In the Euclidean derivation, this creates the frustrum of a cone. In taxicab geometry, the discussion above shows us this will create the frustrum of a taxicab cone, which is equivalent to the frustrum an Euclidean right square pyramid (partially shown in Figure 7.3.2). To find the taxicab surface area of the frustrum, we will compute the Euclidean area and then scale the area by examining the rotations of the frustrum sides to the xy-plane.

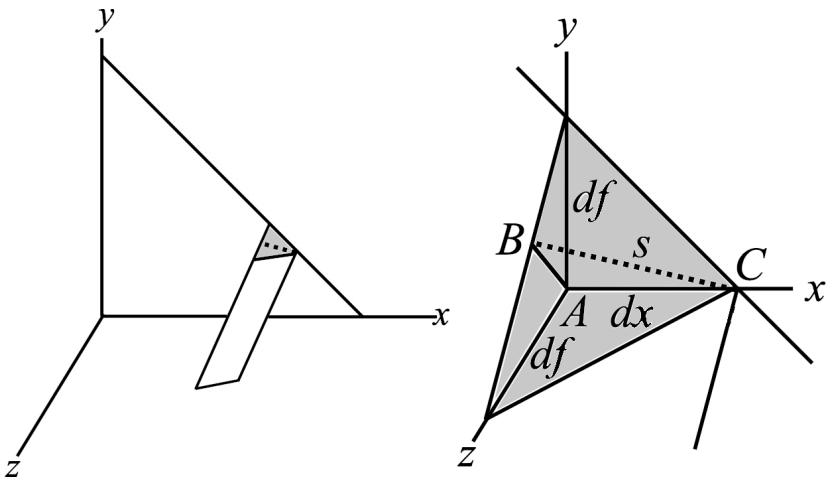


Figure 8.3.2 A portion of the revolution of a line segment about the x-axis in a taxicab manner with a close-up of the shaded triangle. (The full revolution creates the frustrum of an Euclidean right square pyramid.)

The Euclidean surface area of the frustrum of a pyramid is

$$\frac{1}{2}(p_1 + p_2)s$$

where p_1 and p_2 are the perimeters of the "top" and "bottom" edges of the frustrum and s is the slant height. For the k^{th} subinterval, the Euclidean perimeters are $4\sqrt{2}f(x_{k-1})$ and $4\sqrt{2}f(x_k)$. To compute the slant height, we project the shaded triangle in Figure 8.3.2 onto a plane parallel to the yz-plane. The slant height s is the hypotenuse of the right triangle ABC. The projected triangle is an isosceles right triangle with legs of length $|f(x_k) - f(x_{k-1})|$ (labeled df in the figure), so the altitude of this triangle is

$$\frac{\sqrt{2}}{2}|f(x_k) - f(x_{k-1})|$$

The other leg of triangle ABC has length Δx_k (labeled dx in the figure). Therefore, the slant height is

$$s = \sqrt{(\Delta x_k)^2 + \frac{1}{2}[f(x_k) - f(x_{k-1})]^2}$$

So, we have the Euclidean surface area of k^{th} frustrum of the approximation:

$$\begin{aligned} S_{k_e} &= \frac{1}{2}(p_1 + p_2)s \\ &= \frac{1}{2}\left(4\sqrt{2}f(x_{k-1}) + 4\sqrt{2}f(x_k)\right)\sqrt{(\Delta x_k)^2 + \frac{1}{2}[f(x_k) - f(x_{k-1})]^2} \\ &= 2\sqrt{2}\left(f(x_{k-1}) + f(x_k)\right)\sqrt{(\Delta x_k)^2 + \frac{1}{2}[f(x_k) - f(x_{k-1})]^2} \end{aligned}$$

The sides of this frustrum form Euclidean angles of 45° with the xy-plane using a cross-sectional plane parallel to the xz-plane. This gives one scaling factor for the taxicab surface area of $\cos 45^\circ + \sin 45^\circ = \sqrt{2}$. The other scaling factor is dependent on the sum of the cosine and sine of the angle of the linear approximation of the function and the x-axis. This gives another scaling factor of

$$\frac{\Delta x_k + |f(x_k) - f(x_{k-1})|}{\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}}$$

Therefore, using Equation 3.2.1 the taxicab surface area of the k^{th} frustrum of the approximation is

$$S_k = \frac{4(f(x_{k-1}) + f(x_k))(\Delta x_k + |f(x_k) - f(x_{k-1})|)\sqrt{(\Delta x_k)^2 + \frac{1}{2}[f(x_k) - f(x_{k-1})]^2}}{\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}}$$

As the derivation now continues along the same lines as the Euclidean version, the Intermediate Value Theorem and the Mean Value Theorem give us values x_k^* and x_k^{**} such that

$$f(x_k^*) = \frac{1}{2}\left(f(x_k) + f(x_{k-1})\right)$$

and

$$f(x_k) - f(x_{k-1}) = f'(x_k^{**})\Delta x_k$$

Therefore,

$$\begin{aligned} S_k &= \frac{8f(x_k^*)\left(\Delta x_k + |f'(x_k^{**})|\,\Delta x_k\right)\sqrt{(\Delta x_k)^2 + \frac{1}{2}[f'(x_k^{**})\Delta x_k]^2}}{\sqrt{(\Delta x_k)^2 + [f'(x_k^{**})\Delta x_k]^2}} \\ &= \frac{8f(x_k^*)\left(1 + |f'(x_k^{**})|\right)\sqrt{1 + \frac{1}{2}[f'(x_k^{**})]^2}}{\sqrt{1 + [f'(x_k^{**})]^2}}\Delta x_k \\ &= 8f(x_k^*)\left(1 + |f'(x_k^{**})|\right)\sqrt{1 - \frac{[f'(x_k^{**})]^2}{2(1 + [f'(x_k^{**})]^2)}}\Delta x_k \end{aligned}$$

Accumulating the frustrums and taking the limit yields the formula for the taxicab lateral surface area of a solid of revolution:

$$S = \int_a^b 2\pi_t f(x)\left(1 + |f'(x)|\right)\sqrt{1 - \frac{[f'(x)]^2}{2\left(1 + [f'(x)]^2\right)}}\,dx \tag{8.3.1}$$

This is a very interesting formula. The portion outside the radical is very reminiscent of the Euclidean formula with the expected changes in the circle circumference factor $(2\pi_t f(x))$ and the arc length factor $(1 + |f'(x)|)$ due to the taxicab metric. The extra radical represents a scaling factor based on the ratio of the slant height to the linear approximation of the curve. The greater the difference between these two quantities the more the frustrum sides are rotated with respect to the xy-plane thus requiring more scaling. If these two quantities are close, the frustrum sides are rotated very little and therefore require little scaling.

To complete the analysis inspired by Janssen, half of a taxicab sphere of radius r is obtained by revolving the function $f(x) = -x + r$ over the interval $[0, r]$. Doubling this result yields the surface area of a whole taxicab sphere:

$$\begin{aligned} S &= 2\int_0^r 2\pi_t(-x+r)\left(1+|-1|\right)\sqrt{1-\frac{(-1)^2}{2\left(1+(-1)^2\right)}}\,dx \\ &= 4\pi_t\sqrt{3}\int_0^r (-x+r)\,dx \\ &= -2\pi_t\sqrt{3}\,(-x+r)^2\Big|_0^r \\ &= 2\pi_t\sqrt{3}r^2 \end{aligned}$$

Such a taxicab sphere is composed of two Euclidean right square pyramids of height $\frac{r\sqrt{6}}{2}$ and base side length $r\sqrt{2}$. The Euclidean surface area is therefore $4r^2\sqrt{3}$. Since the sides are at 45° angles to the xy-plane in both directions, the taxicab area scaling factors are both $\sqrt{2}$. This gives a taxicab surface area of $8r^2\sqrt{3}$ in perfect agreement with the formula obtained from the solid of revolution.

Cylinders

A cylinder can be formed by revolving a horizontal line $y = r$ about the x-axis. The lateral surface area of a cylinder is therefore

$$\begin{aligned} S &= \int_0^h 2\pi_t r(1+0)\sqrt{1-\frac{0^2}{2\left(1+0^2\right)}}\,dx \\ &= 2\pi_t r\,x\Big|_0^h \\ &= 2\pi_t rh \end{aligned}$$

This is very much reminiscent of the Euclidean formula of circumference of the base times height of the cylinder.

Cones

A cone of radius r and height h can be formed by revolving the line $y = mx$ about the x -axis with $m = r/h$. Therefore, the lateral surface area of a cone is

$$\begin{aligned}
S &= \int_0^h 2\pi_t m\, x(1+m)\sqrt{1-\frac{m^2}{2(1+m^2)}}\, dx \\
&= 2\pi_t m(1+m)\sqrt{\frac{2+m^2}{2+2m^2}}\int_0^h x\, dx \\
&= \pi_t m(1+m)\sqrt{\frac{2+m^2}{2+2m^2}}\, x\Big|_0^r \\
&= \pi_t m(1+m)\sqrt{\frac{2+m^2}{2+2m^2}}h^2 \\
&= \frac{\pi_t r h^2}{h}\left(\frac{h+r}{h}\right)\sqrt{\frac{2+r^2/h^2}{2+2r^2/h^2}} \\
&= \pi_t r(r+h)\sqrt{\frac{2h^2+r^2}{2h^2+2r^2}}
\end{aligned}$$

There are shades of the Euclidean formula for the lateral surface area of a cone in this formula:

$$S = \pi r\left(r + \sqrt{h^2+r^2}\right)$$

Paraboloids

Using the definition of a taxicab parabola from Section 7.4, and limiting to a symmetric horizontal parabola akin to upper half of Figure 7.4.4, Equation 7.4.2 gives the right half of the parabola with focus $(a, 0)$ and directrix $x = -a$:

$$f(x) = \begin{cases} 2x, & 0 \le x \le a \\ 2a, & x > a \end{cases}$$

Restricting the total "height" of the parabola to h and revolving the curve about the x-axis yields an open-top taxicab paraboloid (Figure 8.3.3) with lateral surface area

$$\begin{aligned}
S &= \int_0^a 2\pi_t 2x(1+2)\sqrt{1-\frac{2^2}{2(1+2^2)}}\, dx + \int_a^h 2\pi_t 2a(1+0)\sqrt{1-\frac{0^2}{2(1+0^2)}}\, dx \\
&= 6\pi_t\sqrt{\frac{3}{5}}\, x^2\Big|_0^a + 4\pi_t a\, x\Big|_a^h \\
&= \frac{6\pi_t\sqrt{15}}{5}a^2 + 4\pi_t a(h-a)
\end{aligned}$$

The first term is the lateral surface area of a cone with radius $r = 2a$ and height $h = a$. This is the bottom of the paraboloid. The remainder of the lateral surface area of the paraboloid is a cylinder with radius $r = 2a$ and height $h \to h - a$ which is precisely what is added by the second term.

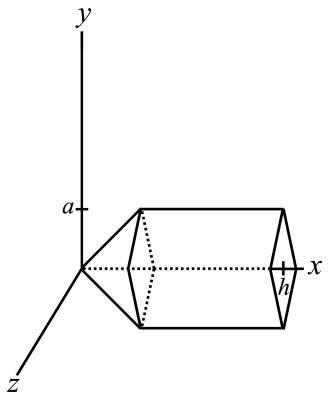


Figure 8.3.3 A taxicab paraboloid

[It should be noted that the original development of paraboloid surface area in Section 5 of the paper "The Nature of Length, Area, and Volume in Taxicab Geometry" was incorrect. The piecewise function used a slope of 1 instead of 2, meaning it generated a sphere as the bottom of the paraboloid rather than a cone.]

Ellipsoids from Two-Foci Ellipses

In Euclidean geometry, there is not a closed form for the surface area an ellipsoid. In taxicab geometry, there are two kinds of ellipses (Section 7.3): two-foci ellipses and focus-directrix ellipses. The two-foci definition leads to a very concise and regular piecewise function definition composed of straight lines, so not only is the computation straightforward but it avoids the closed-form issue of Euclidean geometry. The surface area of a taxicab ellipsoidal solid of revolution based on a two focus ellipse is

$$\begin{aligned}
S &= 2\pi_t \int_{-a}^{b-\frac{s}{2}} \left(x + \frac{s}{2}\right)(1 + |1|)\sqrt{1 - \frac{(1)^2}{2(1+(1)^2)}}\,dx \\
&\quad + 2\pi_t \int_{b-\frac{s}{2}}^{-b+\frac{s}{2}} b(1+0)\sqrt{1 - \frac{(0)^2}{2(1+(0)^2)}}\,dx \\
&\quad + 2\pi_t \int_{-b+\frac{s}{2}}^{a} \left(-x + \frac{s}{2}\right)(1 + |-1|)\sqrt{1 - \frac{(-1)^2}{2(1+(-1)^2)}}\,dx \\
&\quad + 2\left(\frac{1}{2}\pi_t\left(\frac{s}{2} - a\right)^2\right) \\
&= 2\pi_t\sqrt{3}b^2 - 2\pi_t\sqrt{3}\left(\frac{s}{2} - a\right)^2 + 2\pi_t b(s-2b) + \pi_t\left(\frac{s}{2} - a\right)^2
\end{aligned}$$

For a circular ellipsoid (sphere), setting $s = 2a$ and $a = b$ causes this formula to reduce to the sphere result above with $b = r$. For a hexagonal ellipsoid (Figure 7.3.1b), the first term is the surface area of the ends which combine to equal a taxicab sphere of radius b; the second and last terms are zero; and, the third term is the area of a taxicab cylinder of radius b and length $s - 2b$ composing the middle of the ellipsoid. For the octagonal ellipsoid (Figure 7.3.1a), the first term is an overestimate since the top of the sphere is not present on either end. This is corrected by the subtraction of the second term amounting to a taxicab sphere of radius $s/2 - a$. (A similar correction term is seen in the volume formula in Section 8.4.) The last term accounts for the area of the ends of the ellipsoid which are taxicab circles of radius $s/2 - a$.

Section 8.4: Solids of Revolution

As discussed in Section 3.3, we expect the Euclidean and taxicab volume of a solid (a three-dimensional object) in three dimensions to be equal. Only in four dimensions would we expect orientation of the solid to impact the taxicab volume when compared to the Euclidean volume. The main topic to be examined here is when a solid of revolution is created by revolving an area around an axis. With the lessons learned in Section 8.3, such solids will have a different shape than Euclidean solids of revolution given the different in shape of a taxicab circle versus an Euclidean circle.

To compute the volume of a Euclidean solid of revolution, the traditional "slicing" method involves computing the volume of an infinitesimal cylinder as dx multiplied by the area of the Euclidean circle of revolution. This method can be adapted directly into taxicab geometry by replacing the Euclidean circle area with the taxicab circle area *if* the volume of a taxicab cylinder is found in the same manner.

A taxicab cylinder resembles an Euclidean cylinder except that the cross section is a taxicab circle (Figure 8.4.1). For such a cylinder lying along the x-axis, the cross-sectional area will lie in a plane parallel to the yz-plane and therefore the Euclidean and taxicab areas will agree. In addition, the height of the cylinder will agree in the two geometries as it lies along a coordinate axis. Based on our assumption that volume will agree in three dimensions between the geometries, the taxicab volume of a cylinder is the product of the area of the cross-sectional taxicab circle and the height.

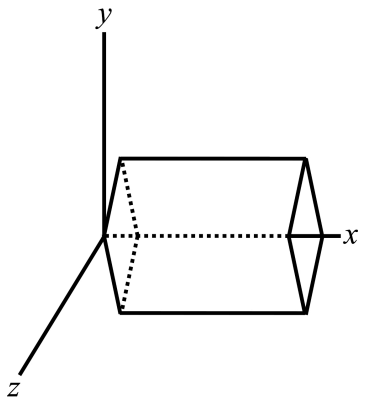


Figure 8.4.1 A taxicab cylinder

A taxicab circle of radius r is composed of four isosceles triangles with area $r^2/2$. Therefore, a taxicab cylinder of radius r and height h will have volume

$$V_t = 2r^2h = \frac{1}{2}\pi_t r^2 h$$

With this information we can now compute the volume of a solid of revolution obtained by revolving a continuous function f about the x-axis over an interval $[a, b]$:

$$V_t = \int_a^b \frac{1}{2}\pi_t [f(x)]^2 \, dx \tag{8.4.1}$$

Continuing our primary example, the upper half of a taxicab circle of radius r centered at the origin is described by

$$f(x) = \begin{cases} x + r & -r \le x \le 0 \\ -x + r & 0 < x \le r \end{cases}$$

Using Equation 8.4.1 the volume of a taxicab sphere obtained by revolving the upper half of the circle about the x-axis is

$$\begin{aligned} V &= \int_{-r}^{r} \frac{1}{2}\pi_t [f(x)]^2 \, dx \\ &= \frac{1}{2}\pi_t \int_{-r}^{0} (x + r)^2 \, dx + \frac{1}{2}\pi_t \int_{0}^{r} (-x + r)^2 \, dx \\ &= \frac{1}{6}\pi_t (x + r)^3 \Big|_{-r}^{0} - \frac{1}{6}\pi_t (-x + r)^3 \Big|_{0}^{r} \\ &= \frac{1}{3}\pi_t r^3 \end{aligned}$$

This result agrees precisely with the volume of a taxicab sphere as a special case of a tetrahedron found by H. Baris Çolakoglu providing some assurance of our concept of volume and computational approach.

It should also be noted that the surface area of a sphere is not the derivative of the volume with respect to the radius. This is a consequence of the radius of the sphere not being everywhere perpendicular to the surface.

Cones

Using the same approach as Section 8.3, a cone of radius r and height h can be formed by revolving the line $y = mx$ about the x-axis with $m = r/h$. Therefore, the volume of a cone is

$$\begin{aligned} V &= \int_{0}^{h} \frac{1}{2}\pi_t \left(f(x)\right)^2 \, dx \\ &= \int_{0}^{h} \frac{1}{2}\pi_t (mx)^2 \, dx \\ &= \frac{m^2}{6}\pi_t x^3 \Big|_{0}^{h} \\ &= \frac{1}{6}\pi_t r^2 h \end{aligned} \tag{8.4.2}$$

As in Euclidean geometry, this is one-third the volume of a taxicab cylinder.

Paraboloids

For the volume of a paraboloid, we use the piecewise parabola function given in Section 8.3 in the solid of revolution volume equation to get

$$\begin{aligned} V &= \int_0^h \frac{1}{2}\pi_t \left(f(x)\right)^2 \, dx \\ &= \int_0^a \frac{1}{2}\pi_t (2x)^2 \, dx + \int_a^h \frac{1}{2}\pi_t (2a)^2 \, dx \\ &= \frac{2}{3}\pi_t x^3 \Big|_0^a + 2\pi_t a^2 x \Big|_a^h \\ &= \frac{2}{3}\pi_t a^3 + 2\pi_t a^2 (h-a) \end{aligned}$$

Again, as expected, this is the sum of the volume of a cylinder of radius $2a$ and height $h-a$ and a cone of radius $2a$ and height a.

[See also the note at the end of the paraboloid development in Section 8.3.]

Ellipsoids from Two-Foci Ellipses

Using the two-foci ellipse piecewise function (Equation 7.3.2) in the solid of revolution volume equation, the ellipsoid solid of revolution will have volume

$$\begin{aligned} V &= \frac{1}{2}\pi_t \int_{-a}^{b-\frac{s}{2}} \left(x + \frac{s}{2}\right)^2 \, dx + \frac{1}{2}\pi_t \int_{b-\frac{s}{2}}^{-b+\frac{s}{2}} b^2 \, dx + \frac{1}{2}\pi_t \int_{-b+\frac{s}{2}}^{a} \left(-x + \frac{s}{2}\right)^2 \, dx \\ &= \frac{1}{3}\pi_t b^3 - \frac{1}{6}(s-2a)^3 + \frac{1}{2}\pi_t b^2 (s-2b) \end{aligned}$$

For the case of a spherical taxicab ellipsoid (derived from the ellipse in Figure 7.3.1c), this formula reduces to

$$V = \frac{1}{3}\pi_t b^3$$

in agreement with our previous result. For the case of a hexagon (ellipse in Figure 7.3.1b), this formula reduces to

$$V = \frac{1}{3}\pi_t b^3 + \frac{1}{2}\pi_t b^2 (2a-2b)$$

which is the sum of the volume of a taxicab sphere of radius b and a taxicab cylinder of radius b and height $2a-2b$. This is to be expected since a hexagonal taxicab ellipsoid is composed of a taxicab cylinder capped by two taxicab half-spheres.

Ellipsoids from Focus-Directrix Ellipses

If the directrix is vertical, a focus-directrix ellipse has a horizontal axis of symmetry. In this case, half of it is suitable for being rotated about the axis to generate a surface or solid of revolution. The solid generated is two cones attached at their bases. As an alternate approach, instead of performing the integration from scratch, we construct the formula based on the result above for the volume of a cone.

Consider a focus-directrix ellipse with focus $F = (0, 0)$ and directrix $x = -c$, $c > 0$, and eccentricity e. If the upper half of the ellipse is revolved around the x-axis, two taxicab cones are produced. Using Equation 7.3.4, it is easy enough to see that both the left cone and right cone will have radius ec, while the left cone has height $\frac{ec}{e+1}$ and the right cone has height $\frac{ec}{1-e}$. Using Equation 8.4.2 for each cone we have

$$\begin{aligned} V &= \frac{1}{6}\pi_t(ec)^2\frac{ec}{e+1} + \frac{1}{6}\pi_t(ec)^2\frac{ec}{1-e} \\ &= \frac{1}{6}\pi_t(ec)^2\left(\frac{ec}{e+1} + \frac{ec}{1-e}\right) \end{aligned}$$

Section 8.5: Derivatives of Trigonometric Functions

From the definition of the taxicab cosine and sine functions in Section 4.3, it is clear that the derivatives of these functions alternate from 0.5 to -0.5 about the extrema. Not only are their derivatives now piecewise constant, the derivative relationship between sine and cosine seen in Euclidean geometry has been lost. Finally, sine and cosine are not everywhere differentiable because of the sharp corners at the extrema.

If we simply take the definition of tangent as the ratio of sine to cosine, use the closed-form expressions in Equation 4.3.2, and differentiate with respect to θ, we obtain the derivative of the taxicab tangent function. As explained above, we will take the proof in two cases: on intervals $[2k-1, 2k)$ with k even and odd. For k even,

$$\begin{aligned}
\frac{d}{d\theta}(\tan\theta) &= \frac{d}{d\theta}\left[\frac{i^{k-2}(k-\frac{1}{2}\theta)}{i^{k-2}(k-1-\frac{1}{2}\theta)}\right] \\
&= \frac{i^{k-2}(k-1-\frac{1}{2}\theta)(-\frac{1}{2}i^{k-2}) - i^{k-2}(k-\frac{1}{2}\theta)(-\frac{1}{2}i^{k-2})}{(i^{k-2}(k-1-\frac{1}{2}\theta))^2} \\
&= \frac{i^{2k-4}(-\frac{1}{2}k+\frac{1}{2}+\frac{1}{4}\theta+\frac{1}{2}k-\frac{1}{4}\theta)}{(i^{k-2}(k-1-\frac{1}{2}\theta))^2} = \frac{\frac{1}{2}(i^2)^k i^{-4}}{(i^{k-2}(k-1-\frac{1}{2}\theta))^2},\ (k \text{ even}) \\
&= \frac{\frac{1}{2}}{(i^{k-2}(k-1-\frac{1}{2}\theta))^2} = \frac{\frac{1}{2}}{\cos^2\theta} = \frac{1}{2}\sec^2\theta
\end{aligned}$$

For k odd, picking up at the 3rd line, we have

$$\begin{aligned}
\frac{d}{d\theta}(\tan\theta) &= \ldots \\
&= \frac{\frac{1}{2}(i^2)^k i^{-2}}{(i^{k-1}(k-\frac{1}{2}\theta))^2},\ (k \text{ odd}) \\
&= \frac{\frac{1}{2}}{(i^{k-1}(k-\frac{1}{2}\theta))^2} = \frac{\frac{1}{2}}{\cos^2\theta} = \frac{1}{2}\sec^2\theta
\end{aligned}$$

So, the derivative of tangent is

$$\frac{d}{d\theta}(\tan\theta) = \frac{1}{2}\sec^2\theta \tag{8.5.1}$$

and we can confirm that the taxicab tangent function is differentiable wherever cosine is non-zero. The factor of $\frac{1}{2}$ is the same as the factor seen in the derivatives of the sine and cosine functions. It is nevertheless a curious factor not seen in the Euclidean derivative of the tangent function.

The derivative of the taxicab cotangent function is found in a similar manner to tangent. The proof will therefore be omitted, but the result is $-\frac{1}{2}\csc^2\theta$. This again is very reminiscent of the derivative of the Euclidean cotangent function with the added factor of $\frac{1}{2}$.

As we saw earlier, the taxicab secant function is expected to not be differentiable where cosine has an extremum. The reason behind this is the fact that cosine is not differentiable at these points. This will prevent us from finding a single closed-form expression for the derivative of secant. But, as with tangent we will take the proof in two cases over the intervals $[2k-1, 2k)$ with k even and odd. Using reciprocals from Equation 4.3.2, for k even we have

$$\begin{aligned}
\frac{d}{d\theta}(\sec\theta) &= \frac{d}{d\theta}\left[\frac{1}{i^{k-2}}\left(k-1-\frac{1}{2}\theta\right)^{-1}\right] \\
&= -\frac{1}{i^{k-2}}\left(k-1-\frac{1}{2}\theta\right)^{-2}\left(-\frac{1}{2}\right) \\
&= \frac{\frac{1}{2}}{i^{k-2}(k-1-\frac{1}{2}\theta)^2} \\
&= \frac{1}{2}\sec\theta\left(\frac{1}{k-1-\frac{1}{2}\theta}\right) \\
&= \frac{1}{2}\sec\theta\left(\frac{k-\frac{1}{2}\theta-(k-1-\frac{1}{2}\theta)}{k-1-\frac{1}{2}\theta}\right) = \frac{1}{2}\sec\theta(\tan\theta-1)
\end{aligned}$$

And, for k odd we have

$$\begin{aligned}
\frac{d}{d\theta}(\sec\theta) &= \frac{d}{d\theta}\left[\frac{1}{i^{k-1}}\left(k-\frac{1}{2}\theta\right)^{-1}\right] \\
&= -\frac{1}{i^{k-1}}\left(k-\frac{1}{2}\theta\right)^{-2}\left(-\frac{1}{2}\right) \\
&= \frac{\frac{1}{2}}{i^{k-1}(k-\frac{1}{2}\theta)^2} \\
&= \frac{1}{2}\sec\theta\left(\frac{1}{k-\frac{1}{2}\theta}\right) \\
&= \frac{1}{2}\sec\theta\left(\frac{1-k+\frac{1}{2}\theta+k-\frac{1}{2}\theta)}{k-\frac{1}{2}\theta}\right) = \frac{1}{2}\sec\theta(\tan\theta+1)
\end{aligned}$$

This is a very interesting result. Shades of the Euclidean derivative are present, but a whole extra secant term is added or subtracted in the taxicab case. As mentioned above, there is no closed-form expression for the derivative of taxicab secant, and the function has a non-differentiable corner point at $4k - 2$ t-radians.

While this presentation of the taxicab secant derivative contrasts with Euclidean geometry nicely, if we apply the Pythagorean-style trig identities for taxicab geometry the secant derivative becomes even more interesting. When k is even, θ lives in quadrants II or IV where tangent is negative. From Equation 4.4.6, in quadrant II $-\tan\theta + 1 = -\sec\theta$ and in quadrant IV, $-\tan\theta + 1 = \sec\theta$. Using these substitutions in the first result above we have the derivative of secant as $\pm\sec^2\theta$. A similar analysis applies to the second result above. Therefore, on intervals $[4k,\ 4k + 4)$ (excluding the asymptote) with k even and odd,

$$\frac{d}{d\theta}(\sec\theta) = \begin{cases} \frac{1}{2}\sec^2\theta, & \text{if } k \text{ is even} \\ -\frac{1}{2}\sec^2\theta, & \text{if } k \text{ is odd} \end{cases} \tag{8.5.2}$$

So, unlike the Euclidean secant function, the taxicab secant function is proportional to the square of itself.

The derivative of cosecant is found in a similar manner to secant. The proof will therefore be omitted, but the result is shown below. Over the interval $[2k - 1, 2k)$ with k even and odd, the derivative of cosecant is

$$\frac{d}{d\theta}(\csc\theta) = \begin{cases} -\frac{1}{2}\csc\theta(\cot\theta - 1), & \text{if } k \text{ is even} \\ -\frac{1}{2}\csc\theta(\cot\theta + 1), & \text{if } k \text{ is odd} \end{cases}$$

And similarly, over the interval $[4k - 2, 4k + 2)$ (excluding the asymptote) with k even and odd, the derivative of cosecant can also be expressed as

$$\frac{d}{d\theta}(\csc\theta) = \begin{cases} -\frac{1}{2}\csc^2\theta, & \text{if } k \text{ is even} \\ \frac{1}{2}\csc^2\theta, & \text{if } k \text{ is odd} \end{cases} \tag{8.5.3}$$

Chapter 9
Miscellaneous Topics

Section 9.1: Parallax

Parallax, the apparent shift of an object due to the motion of the observer, is a commonly used method for estimating the distance to a nearby object. The method of stellar parallax was used extensively to find the distances to nearby stars in the 19th and early 20th centuries. We now wish to explore the method and results of parallax in taxicab geometry and examine how these differ from the Euclidean method and results. We will discover that the taxicab method yields the same formula commonly used as an approximation in the Euclidean case with the exception that the formula is exact in taxicab geometry.

Suppose that as a citizen of Modified Taxicab-land you wish to find the distance to a nearby object Q in the first quadrant, and that there is also a distant reference object P "at infinity" essentially in the same direction as Q with reference angle θ (Figure 9.1.1). The distant reference object should be far enough away so that it appears stationary when you move small distances. We may assume without loss of generality that the object Q does not lie on either axis, for if it did, we could move a small distance to get the object in the interior of the first quadrant.

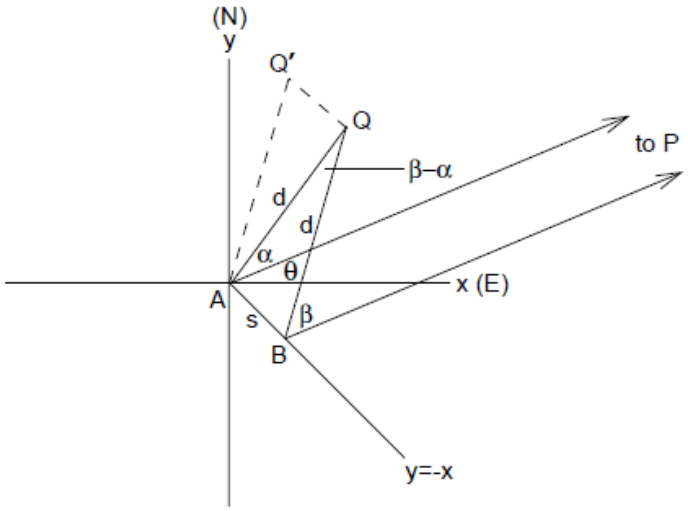


Figure 9.1.1 A parallax diagram in taxicab geometry.

Initially standing at A, measure the angle α between Q and P using the taxicab equivalent of a cross staff or a sextant. Now, for reasons to be apparent later, you should move a small distance (relative to the distance to the object) in such a way that the distance to the object does not change. This can be accomplished by moving in either of two directions, and, provided you move only a small distance, the object remains in the interior of the first quadrant. Furthermore, exactly one of these directions results in the angle between Q and P being increased, so that the situation depicted in Figure 9.1.1 is generic.

You have therefore moved from A to B in one of the following directions: NW, NE, SW, or SE. Now measure the new angle β between the two objects. With this information we can now find the taxicab distance to the object Q. Construct the point Q' such that $\overline{QQ'}$ is parallel to $\overline{AB}$ and $\ell(\overline{QQ'}) = \ell(\overline{AB}) = s$. The angle $\angle PAQ'$ has measure β since it is merely a translation of $\angle QBP$. Thus, $\angle QAQ'$ has measure $\beta - \alpha$. Now, the lengths of $\overline{AQ}$ and $\overline{BQ}$ are equal since the direction of movement from A to B was shrewdly chosen so that the distance to the object remained constant. Since translations do not affect lengths, this implies $\overline{AQ}$ and $\overline{AQ'}$ have equal lengths. Hence, the points Q and Q' lie on a taxicab circle of radius d centered at A. Using the formula for the length of a taxicab arc in Theorem 4.1.4,

$$d = \frac{s}{\beta - \alpha} \tag{9.1.1}$$

where s and d are taxicab distances and $\beta - \alpha$ is a taxicab angle. This formula is identical to the Euclidean distance *estimation* formula with s and d Euclidean distances and $\beta - \alpha$ an Euclidean angle. However, as we shall now see, the commonly used Euclidean version is truly an approximation and not an exact result. This realization is necessary to logically link the commonly used Euclidean formula and the taxicab formula.

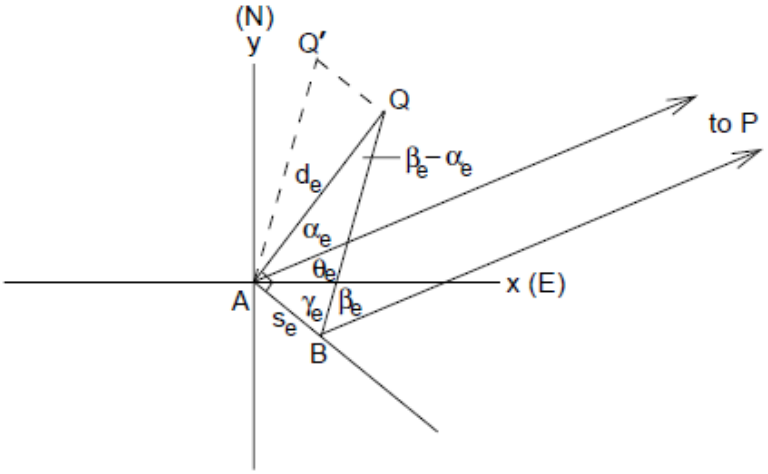


Figure 9.1.2 A parallax diagram in Euclidean geometry.

Using Figure 9.1.2 but with all distances and angles now Euclidean, we isolate $\triangle QAB$. Using the law of sines and the fact that $\gamma = 3\pi/4 - (\beta_e + \theta_e)$, we have

$$d_e = \frac{s_e(\cos_e(\beta_e + \theta_e) + \sin_e(\beta_e + \theta_e))}{\sqrt{2}\sin_e(\beta_e - \alpha_e)} \tag{9.1.2}$$

This formula can be simplified by moving from A to B in a direction perpendicular to the line of sight to Q from A rather than in one of the four prescribed directions above. In this case $m(\angle QBA) = \pi/2 - (\beta_e - \alpha_e)$ (Figure 9.1.2). Thus, the law of sines gives the Euclidean parallax formula

$$d_e = \frac{s_e}{\tan_e(\beta_e - \alpha_e)}$$

If we now apply the approximation $\tan_e(\beta_e - \alpha_e) \approx (\beta_e - \alpha_e)$ for small angles we obtain the commonly used Euclidean parallax formula

$$d_e \approx \frac{s_e}{\beta_e - \alpha_e}$$

which is not exact.

It is interesting to note the quite different movement requirements in the two geometries needed to obtain the best possible approximations of the distance to the object. This difference lies in the methods of keeping the distance to the object as constant as possible. In the Euclidean case, moving small distances on the line tangent to the circle of radius d centered at the object (i.e. perpendicular to the radius of this circle) essentially leaves the distance to the object unchanged. In taxicab geometry, moving in one direction along either $y = x$ or $y = -x$ keeps the distance to the object exactly unchanged.

We are now in a position to justify the link between results 9.1.1 and 9.1.2. Since the line segment $\overline{QQ'}$ of taxicab length s lies on a taxicab circle, $s = \sqrt{2}s_e$. The distance d to the object is given by $d = d_e(\cos_e(\alpha_e + \theta_e) + \sin_e(\alpha_e + \theta_e))$ since the Euclidean angle between the line of sight $\overline{AQ}$ and the x-axis is $(\alpha_e + \theta_e)$. Using Corollary 4.1.2 with $\phi = (\beta_e - \alpha_e)$ and $\psi = (\alpha_e + \theta_e)$, the taxicab measure of $\beta - \alpha$ is given by

$$\beta - \alpha = \frac{2\sin_e(\beta_e - \alpha_e)}{(\cos_e(\beta_e + \theta_e) + \sin_e(\beta_e + \theta_e))(\cos_e(\alpha_e + \theta_e) + \sin_e(\alpha_e + \theta_e))}$$

Using these substitutions, formula 9.1.1 becomes formula 9.1.2.

So, by working in taxicab geometry we have unwittingly discovered an environment in which a parallax method can be used to determine the exact distance to a nearby object rather than just an approximation. Upon deeper reflection, this is ultimately not surprising since there exist directions in which one can travel in taxicab geometry without the distance to an object changing. With this result, and taxicab and Euclidean angle measuring instruments, the exact Euclidean distance to the object can now be found (up to measurement error of course). The trick is to build your own taxicab cross staff or sextant...

Section 9.2: The Butterfly Theorem

In this section we explore the Butterfly Theorem in taxicab geometry. By examining the seemingly narrow circumstances in which the Theorem holds, some light can be shed on why the Theorem always holds in Euclidean geometry.

Theorem 9.2.1 (Euclidean Butterfly Theorem): *Let M be the midpoint of a chord $\overline{PQ}$ of a circle through which two other chords $\overline{AB}$ and $\overline{CD}$ pass, and let X be the intersection of $\overline{AD}$ and $\overline{PQ}$ and Y be the intersection of $\overline{CB}$ and $\overline{PQ}$ (Figure 9.2.1). Then M is also the midpoint of $\overline{XY}$.*

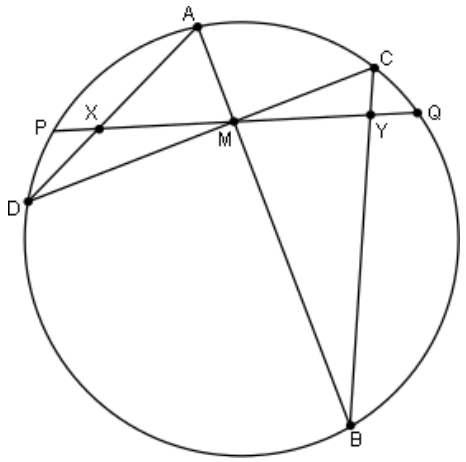


Figure 9.2.1 Classic illustration of the Butterfly Theorem in Euclidean geometry

There are a number of proofs of the Euclidean Butterfly Theorem, but many of them seem to leave the reader asking, "But, WHY is the theorem true?" Let us consider a number of examples of the Theorem in taxicab geometry. We will find out quickly enough that the Theorem fails in general in taxicab geometry, but it might be easiest to begin with very strict assumptions that lead to a positive result and then experiment with removing restrictions to see if the Theorem still holds.

For the first example, assume that the pairs of points A - C, B - D, and P - Q are symmetric about the vertical axis of symmetry of the circle (Figure 9.2.2). With each example, a table is included to facilitate analysis of the details for those interested, Table 9.2.1 in this case.

NOTE: There is another way of expressing the symmetry conditions in the last paragraph that will be important at the end of the examples when we formulate the Theorem for taxicab geometry: the line segments $\overline{AB}$ and $\overline{CD}$ are symmetric about the same line of symmetry as the points P and Q.

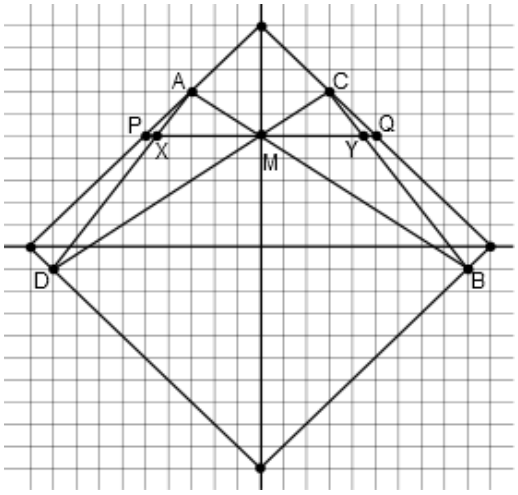


Figure 9.2.2 Successful demonstration of the Butterfly Theorem in taxicab geometry with A - C, B - D, and P - Q symmetric about the vertical axis of symmetry of the circle

Lines			Points		
Line	**Slope**	**Intercept**	**Point**	**x**	**y**
PQ	0	5	M	0	5
$AM\,(AB)$	-2/3	5	B	9	-1
BC	-4/3	11	Y	4.5	5
$CM\,(CD)$	2/3	5	D	-9	-1
AD	4/3	11	X	-4.5	5
XM : 4.5			YM : 4.5		

Table 9.2.1 Details for the successful attempt of the Butterfly Theorem in Figure 9.2.2

To further illuminate this, change to a diagonal line of symmetry (Figure 9.2.3). The theorem once again holds.

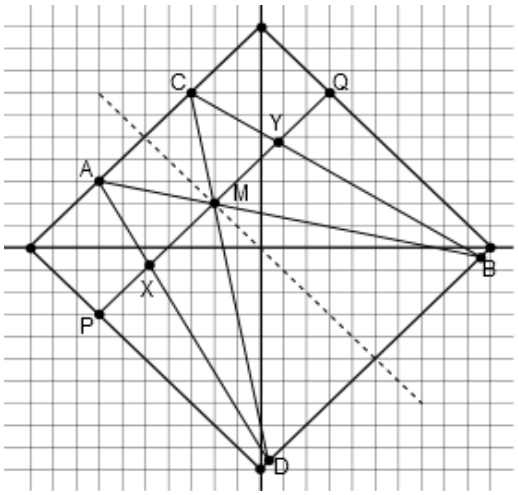


Figure 9.2.3 Successful demonstration of the Butterfly Theorem in taxicab geometry with A - C, B - D, and P - Q symmetric about the same *diagonal* axis of symmetry of the taxicab circle

Lines			Points		
Line	**Slope**	**Intercept**	**Point**	**x**	**y**
PQ	1	4	M	-2	2
$AM\,(AB)$	-1/5	1.6	B	29/3	-1/3
BC	-11/19	100/19	Y	0.8	4.8
$CM\,(CD)$	-5	-8	D	1/3	-29/3
AD	-19/11	-100/11	X	-4.8	-0.8
XM : 5.6			YM : 5.6		

Table 9.2.2 Details for the successful attempt of the Butterfly Theorem in Figure 9.2.3

So, we see that uniform, complete symmetry of the major points about the circle leads to a successful result. Let's begin varying or even taking away aspect of symmetry to see if the result can be retained. First, let's alter the situation in Figure 9.2.3 by having the points P and Q be symmetric about a different line of symmetry for the circle (Figure 9.2.4). This subtly different but still highly symmetric example causes the Theorem to fail.

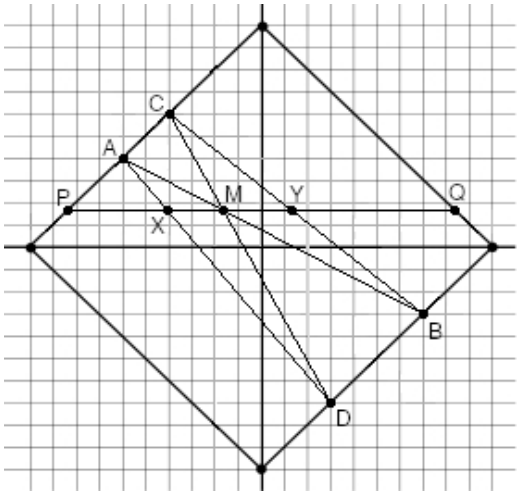


Figure 9.2.4 Failure of the Butterfly Theorem in taxicab geometry when points P and Q are symmetric about a different line of symmetry of the taxicab circle than the line segments $\overline{AB}$ and $\overline{CD}$

Lines			Points		
Line	**Slope**	**Intercept**	**Point**	**x**	**y**
PQ	0	5/3	M	-5/3	5/3
$AM\,(AB)$	-7/13	10/13	B	7	-3
BC	-9/11	30/11	Y	35/27	5/3
$CM\,(CD)$	-13/7	10/7	D	3	-7
AD	-11/9	-10/3	X	-45/11	5/3
$XM : \frac{80}{33}$			$YM : \frac{80}{27}$		

Table 9.2.3 Details for the failure of the Butterfly Theorem attempt illustrated in Figure 9.2.4

What follows are a slew of examples with various levels and kinds of symmetry. They all demonstrate different kinds of failures of the Butterfly Theorem in taxicab geometry. After these examples, we will return to different kind of positive result of the Theorem leading up to a formulation of the Theorem.

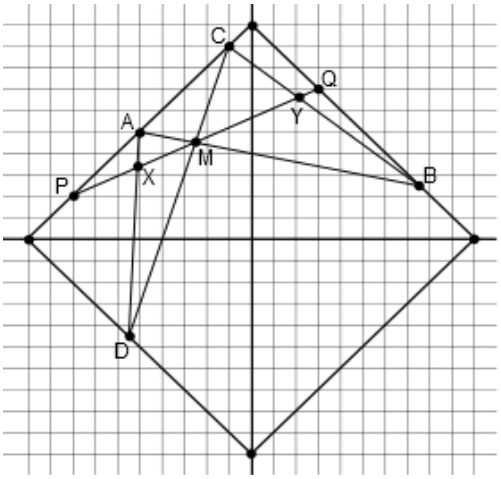


Figure 9.2.5 Failure of the Butterfly Theorem in taxicab geometry with no symmetry present among the major points on the circle

Lines			Points		
Line	**Slope**	**Intercept**	**Point**	**x**	**y**
PQ	5/11	62/11	M	-2.5	4.5
$AM\,(AB)$	-0.2	4	B	7.5	2.5
BC	13/17	140/17	Y	81/38	251/38
$CM\,(CD)$	3	12	D	-5.5	-4.5
AD	19	100	X	-173/34	113/34
$XM: \frac{64}{17}$			$YM: \frac{128}{19}$		

Table 9.2.4 Details for the failure of the Butterfly Theorem attempt illustrated in Figure 9.2.5

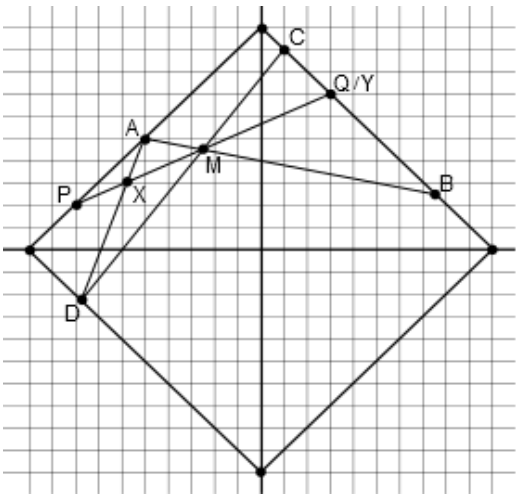


Figure 9.2.6 Failure of the Butterfly Theorem in taxicab geometry with A, C on different circle segments and no symmetry present among the major points on the circle

Lines			Points		
Line	Slope	Intercept	Point	x	y
PQ	5/11	62/11	M	-2.5	4.5
$AM\,(AB)$	-0.2	4	B	7.5	2.5
BC	-1	10	Y	3	7
$CM\,(CD)$	9/7	54/7	D	-7.75	-2.25
AD	29/11	200/11	X	-5.75	133/44
$XM : \frac{52}{11}$			$YM : 8$		

Table 9.2.5 Details for the failure of the Butterfly Theorem attempt illustrated in Figure 9.2.6

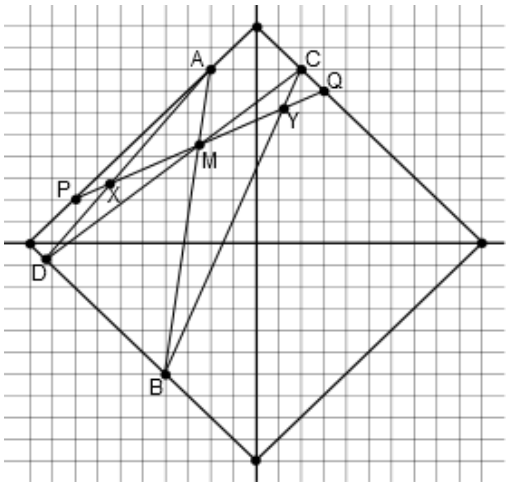


Figure 9.2.7 Failure of the Butterfly Theorem in taxicab geometry with only A and C symmetric on the circle

Lines			Points		
Line	Slope	Intercept	Point	x	y
PQ	5/11	62/11	M	-2.5	4.5
$AM\,(AB)$	7	22	B	-4	-6
BC	7/3	10/3	Y	38/31	192/31
$CM\,(CD)$	7/9	58/9	D	-9.25	-0.75
AD	35/29	302/29	X	-6.35	2.75
$XM : 5.6$			$YM : \frac{168}{31}$		

Table 9.2.6 Details for the failure of the Butterfly Theorem attempt illustrated in Figure 9.2.7

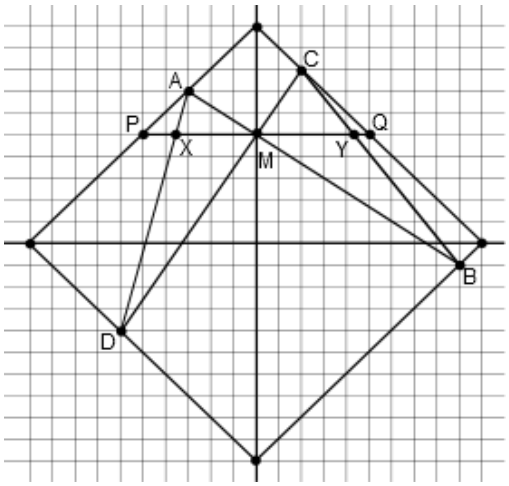


Figure 9.2.8 Failure of the Butterfly Theorem in taxicab geometry with only P and Q symmetric

Lines			Points		
Line	**Slope**	**Intercept**	**Point**	**x**	**y**
PQ	0	5	M	0	5
$AM\,(AB)$	-2/3	5	B	9	-1
BC	-9/7	74/7	Y	13/3	5
$CM\,(CD)$	1.5	5	D	-6	-4
AD	11/3	18	X	-39/11	5
$XM : \frac{39}{11}$			$YM : \frac{11}{3}$		

Table 9.2.7 Details for the failure of the Butterfly Theorem attempt illustrated in Figure 9.2.8

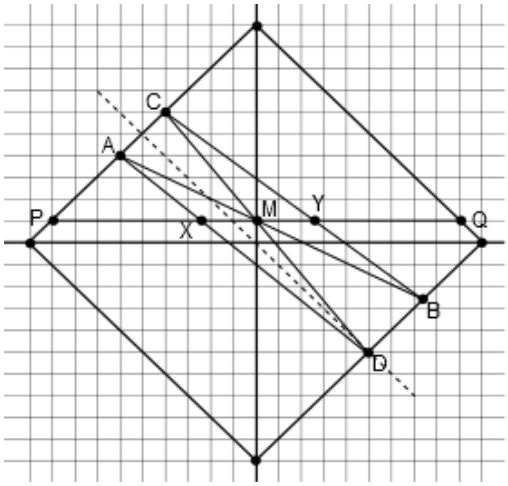


Figure 9.2.9 Failure of the Butterfly Theorem in taxicab geometry with A, C symmetric and P, Q symmetric about different lines of symmetry but with B, D not symmetric

Lines			Points		
Line	**Slope**	**Intercept**	**Point**	**x**	**y**
PQ	0	1	M	0	1
$AM\,(AB)$	-1/2	1	B	22/3	-8/3
BC	-13/17	50/17	Y	33/13	1
$CM\,(CD)$	-1.25	1	D	44/9	-46/9
AD	-41/49	-50/49	X	-99/41	1
$XM: \frac{99}{41}$			$YM: \frac{33}{13}$		

Table 9.2.8 Details for the failure of the Butterfly Theorem attempt in Figure 9.2.9

We now investigate a different kind of positive result of the Butterfly Theorem in taxicab geometry before attempting to present a limited Butterfly Theorem for taxicab geometry. Choose P and Q such that the midpoint M is the *center* of the taxicab circle. Given a choice of P and Q such that the midpoint is the center of the circle, any choice of points A and C will lead to a successful result.

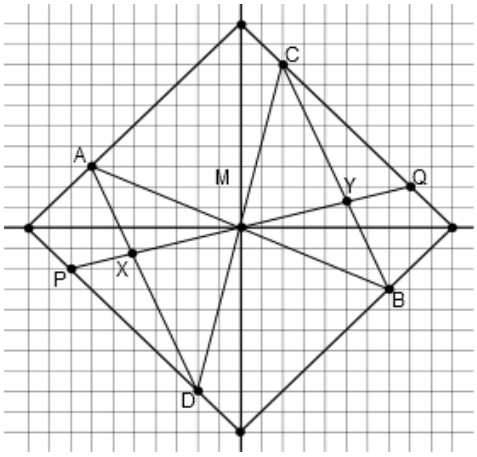


Figure 9.2.10 Successful demonstration of the Butterfly Theorem in taxicab geometry with M at the center of the circle

Lines			Points		
Line	**Slope**	**Intercept**	**Point**	**x**	**y**
PQ	1/4	0	M	0	0
$AM\,(AB)$	-3/7	0	B	7	-3
BC	-2.2	12.4	Y	248/49	62/49
$CM\,(CD)$	4	0	D	-2	-8
AD	-2.2	-12.4	X	-248/49	-62/49
$XM: \frac{310}{49}$			$YM: \frac{310}{49}$		

Table 9.2.9 Details for the successful attempt of the Butterfly Theorem in Figure 9.2.10

This example contains lots of symmetry but in very different ways. Note that the P - Q pair of points are symmetric but now about the center of the circle rather than an axis of symmetry. The A - C and B - D pairs no longer have symmetry, but rather the A - B and C - D pairs have symmetry about the center of the circle. But, note that the line segment $\overline{CD}$ can be obtained from the line segment $\overline{AB}$ by reflecting $\overline{AB}$ across the axis of symmetry $y = x$ and then across the axis of symmetry $y = 0$. So, the line segments are still symmetric, it just takes two reflections to achieve it.

Pulling all of the information from these examples together, we can formulate a limited version of the Butterfly Theorem in taxicab geometry.

Theorem 9.2.2 (Taxicab Butterfly Theorem): *Let M be the midpoint of a chord $\overline{PQ}$ of a circle through which two other chords $\overline{AB}$ and $\overline{CD}$ pass, and let X be the intersection of $\overline{AD}$ and $\overline{PQ}$ and Y be the intersection of $\overline{CB}$ and $\overline{PQ}$. Then M is also the midpoint of $\overline{XY}$ when P and Q are symmetric on the circle and the line segments $\overline{AB}$ and $\overline{CD}$ are symmetric about the circle.*

Just to emphasize, the symmetry make take more than one reflection across axes of symmetry, as in the case of Figure 9.2.10.

Shedding Light on the Euclidean Butterfly Theorem

The early expressions of symmetry at the beginning of this section focused on the point pairs A - C, B - D, and P - Q each being symmetric on the circle. In this formulation, the geometries are in perfect agreement: In Euclidean geometry, there is always an axis of symmetry between any two points on the circle. Despite the most common examples of proof in which this is not clear or explicit, this appears to be the key ingredient to the theorem. Since Euclidean circles are "abundantly symmetric" (every possible diameter is an axis of symmetry) the Euclidean theorem holds for any choice of $\overline{PQ}$, $\overline{AB}$, and $\overline{CD}$ that share the midpoint of $\overline{PQ}$. And, this is the downfall for the theorem in taxicab geometry: taxicab circles have precisely 4 axes of symmetry.

Other aspects to consider are the concepts of similar and congruent triangles. Similar triangles are the crux of many of the proofs of the Euclidean Butterfly Theorem. But, similar triangles are much harder to come by in taxicab geometry. Two triangles must satisfy a SASAS condition in order to conclude similarity.

Very much related to the struggle for similar triangles in taxicab geometry is the lack of preservation of length under rotation for the taxicab metric. Consider the central triangles in Figure 9.2.11. One of the proofs and conclusions of the Euclidean Butterfly Theorem is that two pairs of these triangles are congruent and thus are reflections about the point M (using PQ and its perpendicular as axes for reflection). We could rotate this circle by any degree and the result will remain true. But, this is not the case in taxicab geometry. Only 4 precise rotations will preserve the taxicab length of a line (2, 4, 6, and 8 t-radians), and only reflections about lines parallel to the x-axis, y-axis, and the lines $y = x$ and $y = -x$ will preserve the taxicab length of a line.

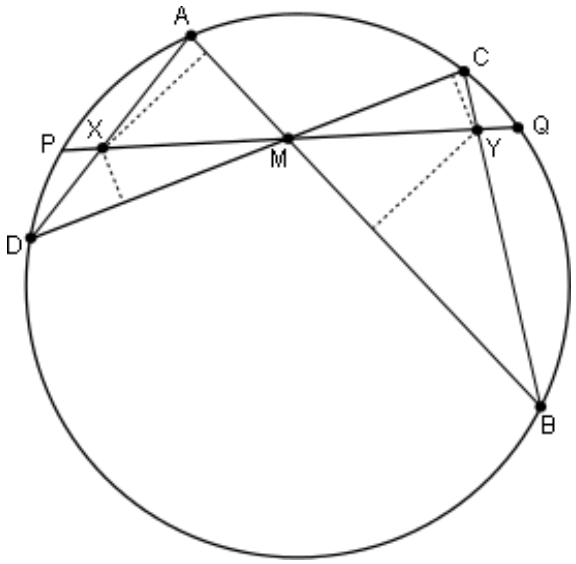


Figure 9.2.11 Detailed illustration of the Butterfly Theorem in Euclidean geometry

If indeed one of the basic conditions in the Butterfly Theorem is for certain triangles to be symmetric about the point M, then the situations in which the theorem holds in Taxicab geometry will be limited to those instances where the axes of reflection about M lie along the 4 axes of symmetry of the taxicab circle, which is what the examples and discussion above seem to imply.

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Kevin received a Bachelor of Science degree in Mathematics (minor in Physics) from Sam Houston State University in 1995 and a Master of Science degree in Mathematics from Oregon State University in 1997. His Master's thesis, supervised by Dr. Tevian Dray, was written in 1996 and published as "Taxicab Angles and Trigonometry" in The Pi Mu Epsilon Journal, Vol. 11, No. 2 (Spring 2000), pp. 87-96. He has had two additional research papers in taxicab geometry published in peer-reviewed mathematical journals. Kevin later passed the first five actuary examinations, each on the first sitting, with three perfect scores.

Kevin has always worked professionally as a self-taught software engineer. He has also worked as a community college mathematics teacher. In addition, Kevin is a genealogist, author, and board game designer and the father of five children.